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arXiv:1110.4864v2 [math-ph] 25 Oct 2011

MATHEMATICAL PHYSICS
PROBLEMS AND SOLUTIONS

The Students Training Contest Olympiad
in Mathematical and Theoretical Physics
(on May 21st – 24th, 2010)

Special Issue № 3 of the Series
«Modern Problems of Mathematical Physics»

Samara
Samara University Press
2010

УДК 51-7+517.958
ББК 22.311
М 34

Authors:
G.S. Beloglazov, A.L. Bobrick, S.V. Chervon, B.V. Danilyuk, M.V. Dolgopolov, M.G. Ivanov, O.G. Panina, E.Yu. Petrova, I.N. Rodionova, E.N. Rykova, M.Y. Shalaginov, I.S. Tsirova, I.V. Volovich, A.P. Zubarev

М 34 Mathematical Physics : Problems and Solutions of The Students Training Contest Olympiad in Mathematical and Theoretical Physics (May 21st – 24th, 2010) / [G.S. Beloglazov et al.]. – Ser. «Modern Problems of Mathematical Physics». – Spec. Iss. № 3. – Samara : Samara University Press, 2010. – 68 p.: il.
ISBN 978-5-86465-494-1

The present issue of the series «Modern Problems in Mathematical Physics» represents the Proceedings of the Students Training Contest Olympiad in Mathematical and Theoretical Physics and includes the statements and solutions of the problems offered to the participants. The contest Olympiad was held on May 21st-24th, 2010 by Scientific Research Laboratory of Mathematical Physics of Samara State University, Steklov Mathematical Institute of Russia's Academy of Sciences, and Moscow Institute of Physics and Technology (State University) in cooperation.

The subjects covered by the problems include classical mechanics, integrable nonlinear systems, probability, integral equations, PDE, quantum and particle physics, cosmology, and other areas of mathematical and theoretical physics.

The present Proceedings is intended to be used by the students of physical and mechanical-mathematical departments of the universities, who are interested in acquiring a deeper knowledge of the methods of mathematical and theoretical physics, and could be also useful for the persons involved in teaching mathematical and theoretical physics.

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ББК 22.311

Editors: B.V. Danilyuk, M.V. Dolgopolov, M.G. Ivanov, I.S. Tsirova, I.V. Volovich

Invited reviewers: M. N. Dubinin, Skobeltsyn Institute of Nuclear Physics of Moscow State University, and Yu. N. Radayev, Institute for Problems in Mechanics of the Russian Academy of Sciences

The Olympiad and the given edition are supported by the grants
ADTP № 3341, 10854 and FTP № 5163
of the Ministry of Education and Science of the Russian Federation,
and by Training and retrainings of specialist center
of Samara State University.

Information support on the website www.labmathphys.samsu.ru/eng

ISBN 978-5-86465-494-1

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Introduction

1 Regulations on The Olympiad

Regulations on holding The Olympiad contest for students on Mathematical and Theoretical Physics were developed in April 2010 [see Special Issue No. 2]. They were signed by the three parties: Samara State University (hereinafter referred to as SamGU), Steklov Mathematical Institute (SMI RAS), and Moscow Institute of Physics and Technology (MIPT). The text of the regulations is given below.

2 Carrying out The Olympiad

On May 21-24th, 2010, All-Russian Student Training Olympiad in Mathematical and Theoretical Physics "Mathematical Physics" with International Participation has been held. It was the second in the series of Olympiads. It is planned that in future such Olympiads will take place annually.

The organizers of the series of Olympiads on Mathematical & Theoretical Physics "Mathematical Physics" are:

Aleksander Anatolyevitch Andreyev (staff member of the Scientific Research Laboratory of Mathematical Physics of SamGU),

Georgiy Sergeyevitch Beloglazov (The University of Dodoma - UDOM, Tanzania; Perm State Pharmaceutical Academy),

Boris Vasilyevitch Danilyuk (staff member of the Scientific Research Laboratory of Mathematical Physics of SamGU),

Mikhail Vyacheslavovitch Dolgopolov (Head of the Scientific Research Laboratory of Mathematical Physics of SamGU),

Vitaliy Petrovitch Garkin (Vice-rector for Academic Affairs, Chairman of the Local Organizing Committee, SamGU),

Mikhail Gennadievich Ivanov (Associate Professor, MIPT),

Yuri Nikolayevitch Radayev (staff member of the Scientific Research Laboratory of Mathematical Physics of SamGU),

Irina Nikolayevna Rodionova (staff member of the Scientific Research Laboratory of Mathematical Physics of SamGU),

Yuri Aleksandrovitch Samarskiy (Deputy Vice Chancellor on Education, MIPT),

Irina Semyonovna Tsirova (docent, SamGU),

Igor Vasilyevitch Volovich (scientific leader of the Scientific Research Laboratory of Mathematical Physics of SamGU, head of the department of Mathematical Physics of MIAN),

Aleksander Petrovitch Zubarev (staff member of the Scientific Research Laboratory of Mathematical Physics of SamGU).

The Olympiad has been held as a team competition. Number of participants of each team – from 3 to 10 students of 2nd to 6th courses (years) of higher educational establishments of Russia, CIS, and other countries. It was allowed that more than one team participates on behalf of any organization. Order of the Olympiad:

The participants have been offered to solve 14 problems. Time to start solving problems of the contest was 11:00 pm Moscow time on May 20th, 2010. The statements of the contest tasks are published in *.pdf format at the webpage of the Olympiad www.labmathphys.samsu.ru/eng/content/view/29/36/ of the website of the Scientific Research Laboratory of Mathematical Physics of SamGU

www.labmathphys.samsu.ru/eng

and have been sent to the registered participants of The Olympiad.

The deadline to send the scanned (or photographed) solutions to the E-mail address of the Mathematical Physics Laboratory: slmp@ssu.samara.ru was 11 pm Moscow time on May 24th, 2010. All participants of The Olympiad who had sent their solutions by E-mail, have received confirmation that their solutions had been accepted.

It was allowed that the participants solve any problems from the number of the proposed ones which they find affordable for the own level of knowledge digestion in different units of mathematics and physics thus participating in the topical scoring nomination (for purpose of this scoring nomination, the problems are aggregated into groups 1 to 3 problems in each).

In the application letter, the name of organization hosting the team should be stated together with the surname, name, for each participant of the team, Department (speciality), course/year; contact E-mail address.

The Nominations of The Olympiad:

1) The overall team scoring based on the three best team participants performance (3 prize-winning team places). In the present Olympiad, it is possible to submit only one solution on behalf of a team; it is advised to mention the author(s) of every solution or solution method [stating also the year(s)/course(s) of studying] at the end of each solution (or method of solution). The winner is the team which participants have solved correctly maximum number of different problems. Any participant of a team has the right to send a solution separately. Within the team scoring, the correct solutions will be considered and accounted. The maximum possible number of points in a team scoring is 14 (because the total number of problems offered is 14).

2) It is possible for a student to participate in the overall personal contest (within the framework of the Olympiad by correspondence) ON CONDITION OF THE

PRESENCE OF AN INDIVIDUAL APPLICATION (REQUEST) from a participant of The Olympiad (3 prize-winning places).

3) Overall team topic scoring (1 – 3 prize-winning places on each subject).

4) Separate team scoring among each of the years (second through sixth courses).

5) Best team among the technical specialities of the institutes of higher education.

6) Other nominations. Separate nomination is supported by the Center on Advanced Training and Professional Development at Samara State University.

The winners of The Olympiad held by correspondence participated in the day competition Olympiad held in Samara in September - 2010 (at the same time with the Second International Conference and School on Mathematical Physics and its Applications). For the above said winners, their travel and/or accommodation expenses were reimbursed.

3 Contents of the problems for the Olympiad contest

The topic range of our 'Olympiad' is related to mathematical methods in describing physical phenomena based on the following units of mathematics and theoretical physics:

theory of differential, integral equations, and boundary-value problems;

theory of generalized functions, integral transform, theory of functions of complex variable;

functional analysis, operational calculus, spectral analysis;

probability theory, theory of random processes;

differential geometry and topology;

theoretical mechanics, electrodynamics, relativity theory, quantum mechanics, and gravitation theory.

New scientific methodological approach to composing the statements of the problems for The Olympiad was first introduced in the sense that about a half of the problems offered to the participants for the solution supposed that certain stage of research (taken from original modern academic research in mathematical physics and its applications) is involved. On the basis of the above mentioned approach, the recommendations on composing statements of the problems for The Olympiad are developed.

In the present issue, we quote the statements of problems offered to the participants of All-Russia Students Training Olympiad in Mathematical and Theoretical Physics "Mathematical Physics" with International Participation (held on May 21-24th, 2010).

4 Results and resume of The Olympiad

In The Olympiad, the teams from the following institutes of higher education and other organizations have participated:

Belarusian State university,

Moscow Institute of Physics and Technology (State University),

National University of Singapore,

Department of Theor. Phys. named after I.E.Tamm of FIAN (the Institute of Physics of Academy of Sciences of Russia) - postgraduate,

Samara State University of Architecture and Construction (two teams),

Samara State Aerospace University (SGAU),

Samara State University,

The Federal University of Siberia,

Ulyanovsk State Pedagogical University, UlGPU (two teams),

The University of Dodoma (UDOM, Tanzania),

Yaroslavl State University (the team of the Physics Department).

The jury has positively assessed the works by the following participants of the teams:

Belarusian State University: Alexey Bobrick.

Moscow Institute of Physics and Technology (State University): Kostjukevich Yury, and the fourth course team: Nikolai Fedotov, Anton Fetisov, Mikhail Shalaginov, Aleksander Shtyk.

Department of Theor. Phys. named after I.E.Tamm of FIAN (the Institute of Physics of Academy of Sciences of Russia): Andrey Borisov.

Samara State University of Architecture and Construction, SGASU (two teams of the students of the 5th year): leader – Sergey Zinakov.

Samara State Aerospace University: Mikhail Malyshev, Yekaterina Pudikova.

Samara State University: team of theoretical physicists – Tatiana Volkova, Matvei Mashchenko, Maksim Nefedov, Yelena Petrova.

The Federal University of Siberia: Artyom Ryasik, Polina Syomina, Anton Sheykin.

Ulyanovsk State Pedagogical University (two teams): 3rd year – Yuri Antonov, Aleksandra Volkova, Oksana Rodionova;

5th year: Maria Vasina, Artyom Ovchinnikov. Aleksander Chaadayev, Aleksander Ernezaks.

The diploma of Laureates or diploma of the winners in nominations have been sent to all above mentioned participants. All participants of The Olympiad have been invited to attend the School-2010 on Applied Mathematical Physics (PMF) from July 1st till July 14th, and the scientific Conference together with another School & Olympiad (August 29th - September 9th, 2010).

The Winners of The Olympiad in the nominations:

The Overall Team Score:

1st Place, 5 problems solved correctly (means, 8 and more points per a problem, maximum 10 points per a problem), the total score is 102 points, - the team of the 4th year of MIPT. The winners are granted the prize - traveling costs be paid for them to participate in the scientific Conference together with School & Olympiad (August 29th - September 7th, 2010).

2nd Place - The Federal University of Siberia

3rd Place - Samara State University

The Total Personal Score:

1st Place, 4 problems solved correctly, score is 126 points, – Alexey Bobrick, Theoretical Physics magistracy at the Department of Physics of Belarusian State University. The winner is granted the prize - either traveling or accommodation costs be paid for him to participate in the scientific Conference together with School & Olympiad (August 29th - September 7th, 2010).

2nd Place, 3 problems solved correctly, score is 74 points, – Yuri Kostyukevitch, the student of the 5th year of the Department of Molecular and Biological Physics, group No. 541, MIPT. Recommended for the Magistracy or (post)graduate school of MIPT.

3rd Place, 2 problems solved correctly, score is 70 points, – Anton Sheykin, the student of the 4th year at Engineering Physical Department of IIFiRE (Physics and Radioelectronics) of Siberian Federal University. Recommended for the Magistracy or (post)graduate school of MIPT.

The best (complete and original) solutions of separate problems: by Alexey Bobrik, Polina Syomina, Mikhail Shalaginov, Anton Sheykin.

1st Team Place among the 3rd year students – SamGU;

2nd Team Place among the 3rd year students – SGAU.

1st Place in Personal contest among the 3rd year students – Maksim Nefedov;

2nd Place in Personal contest among the students of the 3rd year students – Mikhail Malyshev.

1st Place in Personal contest among the 4th year students – Anton Sheykin;

2 – 3 Places in Personal contest among the students of the 4th year – Nikolai Fedotov, Anton Fetisov, and Mikhail Shalaginov.

Among the teams of Pedagogical, Engineering & Technical institutes of higher education:

1st Place – UlGPU, 3 year;

2nd Place – UlGPU, 5 year;

3rd Place – SGASU.

All winners and prize winners of The Olympiad are granted with the free of charge accommodation at the PMF School-2010.

The information on the Olympiad, formulation of the Problems-2010 statements, answers and solutions of the tasks-2010 are presented in this document.

www.labmathphys.samsu.ru/eng slmp@ssu.samara.ru

Organizers of a series of the Mathematical Physics Olympiads: Alexander Andreev, George Beloglazov, Boris Danilyuk, Mikhail Dolgopolov, Vitaliy Garkin, Mikhail Ivanov, Yury Radaev, Irina Rodionova, Yury Samarsky, Irina Tsirova, Igor Volovich, Alexander Zubarev

Problems and Solutions

Statements of the Problems
and Solutions at Students Training Olympiad
on Mathematical & Theoretical Physics
MATHEMATICAL PHYSICS
by Correspondence
with International Participation
May 21st – 24th, 2010

1. Virial for anharmonic oscillations

For a particle moving along the x axis with Hamiltonian

$$H = \frac{p^2}{2m} + \lambda x^{2n},$$

where λ is a positive constant, m is the mass of the particle, p is the momentum of the particle, $n = 1,\ 2,\ 3,\ \ldots$, obtain the relationship between the average values of kinetic $\langle K \rangle$ and potential energy $\langle U \rangle$ using two methods:
(a) directly from the virial theorem (see explanation below);
(b) from the condition

$$\left\langle \frac{d}{dt}(xp) \right\rangle = 0,$$

which is true due to the fact that the motion of the particle is finite.

Instruction: when a particle moves in a potential field, its Hamiltonian H and acting force $\vec{F}$ are defined by

$$H = K + U, \qquad \vec{F} = -\text{grad } U.$$

Explanation to Problem 1. In classical mechanics time average values of kinetic and potential energies of the systems performing finite motion are in rather simple relationship.

The average value for a physical quantity G for a sufficiently large time interval τ is defined in a standard way:

$$\langle G\rangle = \frac{1}{\tau}\int\limits_0^\tau G\ dt.$$

If $\langle K\rangle$ is the average (for a rather long time interval) kinetic energy of the system of point particles (radius-vectors of the particles given as $\vec{r}_i$) subjected to forces $\vec{F}_i$, then the following relation takes place:

$$\langle K\rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i\cdot\vec{r}_i\right\rangle. \qquad (\blacklozenge)$$

The right hand side of equation ($\blacklozenge$) is called Clausius virial, and the equation itself expresses the so called the virial Theorem. The proof of the theorem is given, for example, in [1].

SOLUTION

(a) According to the given statement the particle is moving in the field of a potential force and possesses potential energy $U(x) = \lambda x^{2n}$. The force equals

$$F_x = -\frac{dU}{dx} = -2\lambda n x^{2n-1}.$$

Substitute it into the equation ($\blacklozenge$) which expresses the virial theorem:

$$\langle K\rangle = -\frac{1}{2}\langle\vec{F}\cdot\vec{r}\rangle = -\frac{1}{2}\langle F_x x\rangle = \frac{1}{2}\langle 2\lambda n x^{2n-1}x\rangle = n\langle\lambda x^{2n}\rangle = n\langle U\rangle.$$

(b) According to the given conditions,

$$0 = \left\langle\frac{d}{dt}(xp)\right\rangle = \left\langle\frac{dx}{dt}p + x\frac{dp}{dt}\right\rangle = \left\langle\frac{p}{m}p + xF_x\right\rangle = \left\langle\frac{p^2}{m} + x(-\lambda 2nx^{2n-1})\right\rangle =$$

$$= \left\langle\frac{p^2}{m}\right\rangle - 2n\langle\lambda x^{2n}\rangle = 2\langle K\rangle - 2n\langle U\rangle,$$

that is why

$$\langle K\rangle = n\langle U\rangle.$$

2. Method of successive approximations

Solve the integral Volterra equation of 2nd kind

$$\varphi(x) = \frac{\alpha'(x)}{1-\alpha(x)}\int\limits_0^x \varphi(t)dt + f(x), \qquad (1)$$

where $x \in [0,h]$, $f(x)$ is a given (known) continuous on $[0,h]$ function, $\alpha(x) \in C^1[0,h]$ (continuously differentiable function), and $\alpha(x) \neq 1$, $\alpha'(x)$ is the derivative. Perform your solution check.

Vito Volterra (3 May 1860 — 11 October 1940) was an Italian mathematician and physicist, known for his contributions to mathematical biology and integral equations.

SOLUTION

Using method of successive approximations, we find the solution of the equation (1) through kernel resolvent $K(x,t) = \dfrac{\alpha'(x)}{1-\alpha(x)}$:

$$\varphi(x) = \int_0^x R(x,t)f(t)dt + f(x), \tag{2}$$

$$R(x,t) = \sum_{n=1}^{\infty} K_n(x,t), \tag{3}$$

$$K_1(x,t) = K(x,t), \tag{4}$$

$$K_n(x,t) = \int_t^x K_1(x,s)K_{n-1}(s,t)ds. \tag{5}$$

Using formula (5) we find the repeated kernel $K_2(x,t)$, $K_3(x,t)$

$$K_2(x,t) = \int_t^x \frac{\alpha'(x)}{1-\alpha(x)}\frac{\alpha'(s)}{1-\alpha(s)}ds = \tag{6}$$

$$= \frac{\alpha'(x)}{1-\alpha(x)}[\ln(1-\alpha(t)) - \ln(1-\alpha(x))],$$

$$K_3(x,t) = \frac{\alpha'(x)}{1-\alpha(x)}\int_t^x \frac{\alpha'(s)}{1-\alpha(s)}[\ln(1-\alpha(t)) - \ln(1-\alpha(s))]ds = \tag{7}$$

$$= \frac{\alpha'(x)}{1-\alpha(x)} \left[\frac{\ln^2(1-\alpha(t))}{2} - \ln(1-\alpha(t)) \ln(1-\alpha(x)) + \frac{\ln^2(1-\alpha(x))}{2} \right] =$$

$$= \frac{\alpha'(x)}{2!(1-\alpha(x))} [\ln(1-\alpha(t)) - \ln(1-\alpha(x))]^2.$$

Similarly, using the formula (5),

$$K_4(x,t) = \frac{\alpha'(x)}{3!(1-\alpha(x))} [\ln(1-\alpha(t)) - \ln(1-\alpha(x))]^3, \tag{8}$$

we come to the conclusion that

$$K_n(x,t) = \frac{\alpha'(x)}{(n-1)!(1-\alpha(x))} \ln^{n-1} \left(\frac{1-\alpha(t)}{1-\alpha(x)} \right). \tag{9}$$

Expression (9) should be substituted into the formula (3):

$$R(x,t) = \sum_{n=1}^{\infty} \frac{\alpha'(x)}{1-\alpha(x)} \frac{\ln^{n-1} \left(\frac{1-\alpha(t)}{1-\alpha(x)} \right)}{(n-1)!}. \tag{10}$$

Performing the transformation $n-1=m$ in (10) and recalling the expansion

$$e^z = \sum_{m=0}^{\infty} \frac{z^m}{m!}, \tag{11}$$

we obtain

$$R(x,t) = \frac{\alpha'(x)(1-\alpha(t))}{(1-\alpha(x))^2}. \tag{12}$$

Substituting (12) into formula (2), we obtain

$$\varphi(x) = f(x) + \frac{\alpha'(x)}{(1-\alpha(x))^2} \int_0^x f(t)[1-\alpha(t)]dt. \tag{13}$$

Checking. Let us show that the function (13) is the solution of eq. (1). Designate

$$J(x) = \varphi(x) - \frac{\alpha'(x)}{1-\alpha(x)} \int_0^x \varphi(t)dt, \tag{14}$$

and substitute the function (13) into the right side of equation (14). As a result, we shall obtain

$$J(x) = f(x) + \frac{\alpha'(x)}{[1-\alpha(x)]^2} \int_0^x f(t)[1-\alpha(t)]dt - \tag{15}$$

$$-\frac{\alpha'(x)}{1-\alpha(x)}\int\limits_0^x f(t)dt-\frac{\alpha'(x)}{1-\alpha(x)}\int\limits_0^x dt\int\limits_0^t\frac{\alpha'(t)(1-\alpha(s))f(s)ds}{[1-\alpha(t)]^2}.$$

In the last term of formula (15) let us change the integration order and calculate the inner integral:

$$\int\limits_s^x\frac{\alpha'(t)dt}{[1-\alpha(t)]^2}=\frac{1}{1-\alpha(x)}-\frac{1}{1-\alpha(s)}. \tag{16}$$

The result should be substituted into the formula (15):

$$J(x)=f(x)+\frac{\alpha'(x)}{[1-\alpha(x)]^2}\int\limits_0^x f(t)[1-\alpha(t)]dt- \tag{17}$$

$$-\frac{\alpha'(x)}{1-\alpha(x)}\int\limits_0^x f(t)dt-\frac{\alpha'(x)}{(1-\alpha(x))^2}\int\limits_0^x(1-\alpha(s))f(s)ds+\frac{\alpha'(x)}{1-\alpha(x)}\int\limits_0^x f(s)ds\equiv f(x).$$

Our checking has shown that the function (13) is the correct solution of the equation (1).

3. Evaluation for ultrametric diffusion

When solving equations of the ultrametric diffusion type (that have a relation to the description of conformational dynamics of complicated systems such as biomacromolecules) the results can often be presented in the form of series of exponents. Two of such series are represented below:

$$R(t)=\sum_{n=0}^{\infty}a^{-n}e^{-b^{-n}t},\qquad S(t)=\sum_{n=1}^{\infty}\frac{1}{n^k}a^{-n}e^{-b^{-n}t}.$$

Here t is time, $R(t)$ and $S(t)$ are probabilities that a system is in some definite groups of states, k is some integer number, $a>1$, $b>1$ are some parameters.

Study the asymptotic behavior of functions $R(t)$ and $S(t)$ at $t\to\infty$ and evaluate, if possible, their asymptotics using elementary functions depending on t.

SOLUTION

Let us explore $S(t)$ and $R(t)=S(t)|_{k=0}+e^{-t}$. Note that the function $\frac{1}{x^k}a^{-x}$ decreases, while the function $e^{-b^{-x}t}$ increases with the growth of x. Then in the interval $x-1\le n\le x$ the inequality takes place

$$\frac{1}{x^k}a^{-x}e^{-b^{-(x-1)}t}\le\frac{1}{n^k}a^{-n}e^{-b^{-n}t}\le\frac{1}{(x-1)^k}a^{-(x-1)}e^{-b^{-x}t}$$

takes place. Integrating it with respect to x from n to $n+1$ gives (for $n>1$):

$$a^{-1}\int_n^{n+1}\frac{1}{x^k}a^{-(x-1)}e^{-b^{-(x-1)}t}dx\le\frac{1}{n^k}a^{-n}e^{-b^{-n}t}\le a\int_n^{n+1}\frac{1}{(x-1)^k}a^{-x}e^{-b^{-x}t}dx.$$

Now, by summing over n from 2 to ∞, we obtain:

$$\tilde{S}_{\min}(t)\leqslant\tilde{S}(t)\leqslant\tilde{S}_{\max}(t)$$

where $\tilde{S}(t)\equiv\sum\limits_2^\infty\frac{1}{n^k}a^{-n}e^{-b^{-n}t}$,

$$\tilde{S}_{\min}(t)\equiv a^{-1}\int_2^\infty\frac{1}{x^k}a^{-(x-1)}e^{-b^{-(x-1)}t}dx,\qquad\tilde{S}_{\max}(t)\equiv a\int_2^\infty\frac{1}{(x-1)^k}a^{-x}e^{-b^{-x}t}dx.$$

By switching to new variables, we have:

$$\tilde{S}_{\min}(t)=a^{-1}(\ln b)^{k-1}(\ln t)^{-k}t^{-\frac{\ln a}{\ln b}}\int_0^{b^{-1}t}\left(1-\frac{\ln(b^{-1}y)}{\ln t}\right)^{-k}y^{\frac{\ln a}{\ln b}-1}e^{-y}dy,$$

$$\tilde{S}_{\max}(t)=a(\ln b)^{k-1}(\ln t)^{-k}t^{-\frac{\ln a}{\ln b}}\int_0^{b^{-2}t}\left(1-\frac{\ln(by)}{\ln t}\right)^{-k}y^{\frac{\ln a}{\ln b}-1}e^{-y}dy.$$

Let us designate the function

$$\gamma^{(k)}(z,t,\alpha,\beta)\equiv\int_0^{\alpha t}\left(1-\frac{\ln\beta y}{\ln t}\right)^{-k}y^{z-1}e^{-y}dy,$$

considering $\alpha\beta<1$ for convergence.

Note that the limit for this function is the Gamma-function (see the proof in [2]):

$$\lim_{t\to+\infty}\gamma^{(k)}(z,t,\alpha,\beta)=\int_0^\infty y^{z-1}e^{-y}dy=\Gamma(z).$$

Then at $t\gg1$ it is possible to write

$$\tilde{S}_{\min}(t)=a^{-1}(\ln b)^{k-1}(\ln t)^{-k}t^{-\frac{\ln a}{\ln b}}\Gamma\left(\frac{\ln a}{\ln b}\right)(1+o(t)),$$

$$\tilde{S}_{\max}(t)=a(\ln b)^{k-1}(\ln t)^{-k}t^{-\frac{\ln a}{\ln b}}\Gamma\left(\frac{\ln a}{\ln b}\right)(1+o(t)).$$

Notation $o(t)$ means that in the limit at $t\to\infty$ the value of $o(t)$ tends to zero.

Since

$$S(t)=\tilde{S}(t)+a^{-1}e^{-b^{-1}t},$$

the final asymptotic evaluation is of the form $(\ln t)^{-k}t^{-\frac{\ln a}{\ln b}}$:

$$a^{-1}(\ln b)^{k-1}\Gamma\left(\frac{\ln a}{\ln b}\right)(\ln t)^{-k}t^{-\frac{\ln a}{\ln b}}(1+o(t))\le S(t)\le$$

$$\leq a\,(\ln b)^{k-1}\,\Gamma\left(\frac{\ln a}{\ln b}\right)(\ln t)^{-k}\,t^{-\frac{\ln a}{\ln b}}\,(1+o(t))\,.$$

Because

$$R(t)=\tilde{S}(t)|_{k=0}+e^{-t}+a^{-1}e^{-b^{-1}t},$$

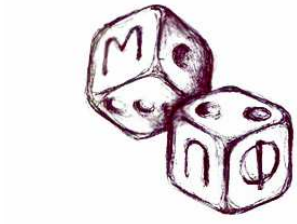
we have also

$$a^{-1}(\ln b)^{-1}\Gamma\left(\frac{\ln a}{\ln b}\right)t^{-\frac{\ln a}{\ln b}}\,(1+o(t))\leq R(t)\leq a(\ln b)^{-1}\Gamma\left(\frac{\ln a}{\ln b}\right)t^{-\frac{\ln a}{\ln b}}\,(1+o(t))\,.$$

4. Double effort

Solve the Volterra integral equation

$$\varphi(x)=x+\int_0^x(s-x)\varphi(s)ds. \tag{1}$$



SOLUTION

The considered equation is in fact Volterra integral equation of 2^{nd} kind with continuous kernel. According to the theory of linear integral equations, it has a unique solution. To find its solution, we reduce it to Cauchy problem for ordinary differential equation. [1]

Let us assume that $\varphi(x)$ is the solution of the equation (1). Double differentiation of the identity equation (1) by x gives:

$$\varphi'(x)=1+\int_0^x(-\varphi(s))\,ds, \tag{2}$$

$$\varphi''(x)=-\,\varphi(x). \tag{3}$$

$$\varphi(x)=C_1\sin x+C_2\cos x. \tag{4}$$

Assuming $x=0$ results in:

$$\varphi(0)=0. \tag{5}$$

[1] Special Issue No. 4 contains also another methods of solution (see [2]).

At $x = 0$ we have from eq. (2):

$$\varphi'(0) = 1. \tag{6}$$

Hence $C_1 = 1$, $C_2 = 0$, and

$$\varphi(x) = \sin x. \tag{7}$$

It is possible to perform a check:

$$J(x) = x + \int_0^x (s - x) \sin s \, ds = \tag{8}$$

$$= x - (s - x) \cos s|_0^x + \int_0^x \cos s \, ds, \tag{9}$$

$$= x - x + \sin(s)|_0^x = \sin x.$$

It is possible to propose other ways to solve this equation, such as with the help of Laplace transformation, or by building a resolvent of the kernel by successive approximations method. However, the technique developed above is the simplest.

5. Random walk

A particle performs random walk on one-dimensional lattice situated on the OX axis, the nodes of the lattice have the coordinates $m = 0, \pm 1, \pm 2, ...$ At the initial time moment $t_0 = 0$ the particle is at the origin of the coordinates. At random time moments t_1, t_2, t_3,... the particle performs the jumps into adjacent lattice nodes with the probabilities of a jump leftwards and rightwards equal to $\dfrac{\alpha}{2}$, the probability of remaining still being $\beta = 1 - \alpha$. The time intervals between the jumps $t_{i+1} - t_i$, $i = 0, 1, 2, ...$ are independent random quantities which have the same exponential distribution $\Phi(t) = \dfrac{1}{\tau} \exp(-t/\tau)$ with expectation τ. Find:

(a) dispersion of the location of the particle as time function t;

(b) probability that the particle is in m-th node at time moment t.

SOLUTION

1. Let us find the probability $p(t, n)$ that the particle during time interval $(0, t]$ would perform exactly n jumps, taking into account that this probability is the distribution function of the Poisson process. Let $t_1, t_2, ...$ be time instants of the jumps, so that

$$t_0 = 0 < t_1 < t_2 < ... < t_{n-1} < t_n < t < t_{n+1}. \tag{1}$$

Probability $p(t, n)$ that the particle during the time interval $(0, t]$ would perform exactly n jumps, can be represented as

$$p(t, n) = \mathrm{M}\left[I\left(t_n < t < t_{n+1}\right)\right], \tag{2}$$

where $\mathrm{M}\left[...\right]$ is expectation and

$$I\left(t_n < t < t_{n+1}\right) = \begin{cases} 1, & \text{if } t_n < t < t_{n+1}, \\ 0, & \text{if } t_n \geq t \quad \text{or} \quad t \geq t_{n+1}. \end{cases} \tag{3}$$

Let us perform Laplace transformation of the function $p(t,n)$:

$$\hat{p}(s,n) = \int_0^\infty dt e^{-st} p(t,n) = \mathrm{M}\left[\int_0^\infty dt e^{-st} I\left(t_n < t < t_{n+1}\right)\right] =$$

$$= \mathrm{M}\left[\frac{e^{-st_n} - e^{-st_{n+1}}}{s}\right]. \tag{4}$$

Due to the fact that $t_n = \sum_{i=1}^{n} \tau_i$ is the sum of independent random variables, then

$$\mathrm{M}\left[e^{-st_n}\right] = \mathrm{M}\left[\exp\left(-s\sum_{i=1}^{n}\tau_i\right)\right] = \prod_{i=1}^{n} \mathrm{M}\left[\exp\left(-s\tau_i\right)\right] = \tag{5}$$

$$= \prod_{i=1}^{n} \int_0^\infty d\tau_i \exp\left(-s\tau_i\right) \Phi(\tau_i) = \hat{\Phi}^n(s),$$

where $\hat{\Phi}(s) = \int_0^\infty d\tau \exp\left(-s\tau\right) \Phi(\tau)$ is Laplace image of $\Phi(\tau)$. From this,

$$\hat{p}(s,n) = \mathrm{M}\left[\frac{e^{-st_n} - e^{-st_{n+1}}}{s}\right] = \hat{\Phi}^n(s)\frac{1-\hat{\Phi}(s)}{s}. \tag{6}$$

Due to the fact that $\Phi(t) = \dfrac{1}{\tau}\exp(-t/\tau)$, it follows that

$$\hat{\Phi}(s) = \frac{1/\tau}{s + 1/\tau}, \tag{7}$$

and

$$\hat{p}(s,n) = \frac{(1/\tau)^n}{(s+1/\tau)^{n+1}}. \tag{8}$$

Transforming from Laplace image to the original, we obtain the distribution function for the Poisson process

$$p(t,n) = \frac{t^n e^{-t/\tau}}{\tau^n n!}. \tag{9}$$

2. Let us find the dispersion $D(t)$ of the location of the particle as a function of time t. Location of the particle after n jumps is defined by the random variable

$X_n(t) \equiv \xi_1 + \xi_2 + ... + \xi_n$, where ξ_i, $i = 1, ..., n$ are independent random variables, possessing the values ± 1 with the probability $\frac{\alpha}{2}$ and 0 with the probability $\beta = 1 - \alpha$. Due to the fact that $\mathrm{M}\left[(\xi_i)^2\right] = \alpha$, $\mathrm{M}(\xi_i) = 0$ and $\mathrm{M}\left[\xi_i \xi_j\right] = 0$ for $i \neq j$, the dispersion equals to

$$D(t) = \sum_{n=0}^{\infty} p(t,n) \mathrm{M}\left[(X_n(t))^2\right] = \sum_{n=0}^{\infty} p(t,n) \alpha n. \tag{10}$$

Substituting into this formula the expression for $p(t,n)$, we obtain

$$D(t) = \sum_{n=1}^{\infty} \frac{t^n e^{-t/\tau}}{\tau^n n!} \alpha n = \alpha e^{-t/\tau} \frac{t}{\tau} \sum_{n=1}^{\infty} \frac{t^{n-1}}{\tau^{n-1}(n-1)!} = \alpha \frac{t}{\tau}. \tag{11}$$

3. Let us find the probability that the particle is located at node m at time moment t. Let $h_n(m)$ designate the probability that the particle is located at the point with coordinate m after n jumps (transitions). Then the probability $f(m,t)$ that the particle is at location m at time moment t will be given by formula

$$f(m,t) = \sum_{n=0}^{\infty} p(t,n) h_n(m). \tag{12}$$

Function $h_n(m)$ equals to

$$h_n(m) = \mathrm{M}\left[\delta_{m,\xi_1+\xi_2+...+\xi_n}\right], \tag{13}$$

where $\delta_{m,n}$ is the Kronecker delta. Due to the fact that $\delta_{m,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\varphi} d\varphi$, it is possible to write

$$h_n(m) = \frac{1}{2\pi} \mathrm{M}\left[\int_{-\pi}^{\pi} e^{i(m-\xi_1-\xi_2-...-\xi_n)\varphi} d\varphi\right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\varphi} \prod_{j=1}^{n} \mathrm{M}\left[e^{-i\xi_j\varphi}\right] d\varphi. \tag{14}$$

Note that

$$\mathrm{M}\left[e^{-i\xi_j\varphi}\right] = \frac{\alpha}{2}\left(e^{-i\varphi} + e^{i\varphi}\right) + 1 - \alpha = \alpha \cos\varphi + 1 - \alpha. \tag{15}$$

Therefore

$$h_n(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\varphi} (\alpha \cos\varphi + 1 - \alpha)^n \, d\varphi. \tag{16}$$

Substituting this equation for $h_n(m)$, and earlier found expression for $p(t,n)$ into the formula for $f(m,t)$, we obtain

$$f(m,t) = \sum_{n=0}^{\infty} \frac{t^n e^{-t/\tau}}{\tau^n n!} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\varphi} (\alpha \cos\varphi + 1 - \alpha)^n \, d\varphi = \tag{17}$$

$$= \frac{1}{2\pi} e^{-t/\tau} \int_{-\pi}^{\pi} e^{im\varphi} d\varphi \sum_{n=0}^{\infty} (\alpha \cos\varphi + 1 - \alpha)^n \frac{t^n}{\tau^n n!} =$$

$$= \frac{1}{2\pi} e^{-t/\tau} \int_{-\pi}^{\pi} e^{im\varphi} e^{t(\alpha\cos\varphi+1-\alpha)/\tau} d\varphi =$$

$$= \frac{1}{2\pi} e^{-t/\tau} \left(\int_0^{\pi} e^{im\varphi} e^{t(\alpha\cos\varphi+1-\alpha)/\tau} d\varphi + \int_0^{\pi} e^{-im\varphi} e^{t(\alpha\cos\varphi+1-\alpha)/\tau} d\varphi \right) =$$

$$= \frac{1}{\pi} e^{-t/\tau} \int_0^{\pi} \cos(m\varphi) e^{t(\alpha\cos\varphi+1-\alpha)/\tau} d\varphi = \frac{1}{\pi} e^{-\alpha t/\tau} \int_0^{\pi} \cos(m\varphi) e^{t\alpha\cos\varphi/\tau} d\varphi.$$

The last equation can be reproduced in a different form, using integral representation of Bessel function $J_m(z)$:

$$J_m(z) = \frac{i^{-m}}{\pi} \int_0^{\pi} \cos(m\varphi)\, e^{iz\cos\varphi} d\varphi. \tag{18}$$

As a result, we obtain

$$f(m,t) = i^m e^{-\alpha t/\tau} J_m\left(-i\frac{\alpha t}{\tau}\right). \tag{19}$$

Taking into consideration that

$$J_m(iz) \equiv i^m I_m(z), \tag{20}$$

where

$$I_m(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+m}}{k!\,(k+m)!} \tag{21}$$

are the modified Bessel functions, we write the final result for the probability of location of the particle at node m at time moment t:

$$f(m,t) = (-1)^m e^{-\alpha t/\tau} I_m\left(-\frac{\alpha t}{\tau}\right) = e^{-\alpha t/\tau} I_m\left(\frac{\alpha t}{\tau}\right). \tag{22}$$

Asymptotics of $f(m,t)$ at $t \to \infty$ with the asymptotic behavior of modified Bessel functions

$$I_m(z) = \frac{e^z}{\sqrt{2\pi z}} \left(1 + O\left(z^{-1}\right)\right) \qquad \text{at} \quad z \to \infty \tag{23}$$

considered, is of the form:

$$f(m,t) = e^{-\alpha t/\tau} \frac{e^{\alpha t/\tau}}{\sqrt{2\pi\alpha t/\tau}} \left(1 + O\left(t^{-1}\right)\right) = \frac{1}{\sqrt{2\pi\alpha t/\tau}} \left(1 + O\left(t^{-1}\right)\right). \tag{24}$$

6. Thermal equations of the Universe evolution

It is assumed that at high temperature (at early stage of the evolution of the Universe) it is possible to describe matter using field theory. Equation of state with good approximation corresponds to ideal quantum gas of massless particles (in the general case, it can be a mixture of ideal Bose- and Fermi-gases). In this theory, under the condition that the temperature T is far from mass threshold yet (radiation[2] dominance, $\rho = 3p$), thermodynamic functions are given by the formulae:

$$\rho = 3p = \frac{\pi^2}{30} N(T) T^4, \tag{1}$$

$$s = \frac{2\pi^2}{45} N(T) T^3, \tag{2}$$

where $N(T)$ is the function related to the number of bosonic and fermionic degrees of freedom ($N(T) = N_b(T) + \frac{7}{8} N_f(T)$), ρ and p are equilibrium energy density and pressure of the matter, s is specific entropy. All expressions are written in the unified system of units $c = \hbar = 1$.

Formulate the dynamic equations of the evolution of the Universe in terms of temperature.

Note. The required equations are not Einstein's equations in the standard form (see Einstein equations, for example, in [3, 4]). It is proposed to write the equations of the evolution of the Universe in Friedmann model using thermodynamic functions and temperature as function of time. It is possible to do this with the use of energy conservation law and condition of adiabatic expansion of the Universe in the framework of the standard cosmologic model.

Instruction. Introduce auxiliary function

$$\epsilon(T) = \frac{k}{a^2 T^2}, \tag{3}$$

where $k = 0, \pm 1$ respectively for flat, open and closed models of the Universe with time-dependent scale factor of $a \equiv a(t)$.

[2]By radiation here, we mean any relativistic object, including relativistic matter as well as photons.

Guth, Alan Harvey – American physicist and cosmologist who has first proposed the idea of cosmological inflation. In 2004, Guth together with Andrew Linde were awarded cosmological prize named after Peter Gruber for his work on the theory of inflation Universe.

SOLUTION

This problem of finding the time dependence of the temperature of the Universe was first formulated by A. Guth in [5]. To solve this problem, we would need to rewrite Einstein–Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho \tag{4}$$

in terms of temperature [4] (here G is the gravitation constant). It is considered that stress energy tensor of the Universe takes the form of energy momentum tensor for ideal liquid [3]. We shall use the energy conservation law

$$\frac{d}{dt}(\rho a^3) = -\,p\frac{d}{dt}(a^3) \tag{5}$$

(the change in energy in a comoving volume element, $d\,(\rho a^3)$, is equal to minus the pressure times the change in volume, $-p\,d\,(a^3)$) and the fact that in the standard cosmological model it is supposed that the Universe undergoes adiabatic expansion

$$\frac{d}{dt}(sa^3) = 0 \tag{6}$$

(the entropy per comoving volume element remains constant).

Let us write Einstein–Friedmann equation (4) in terms of temperature supposing that temperature value is far from the mass threshold (see e. g. [3]). We deal with matter which is found in thermodynamical equilibrium at almost all the time stages during cosmological expansion, so the chemical potential is considered to be zero.

Taking into consideration (1) and auxiliary function (3) let us represent eq. (4) in the form

$$\left(\frac{\dot{a}}{a}\right)^2 + \epsilon(T)T^2 = \frac{4\pi^3}{45} GN(T)T^4. \tag{7}$$

Taking into account equation of state $\rho = 3p$, from the energy conservation law (5) we obtain the relationship

$$\frac{\dot{a}}{a} = -\frac{1}{4}\frac{\dot{\rho}}{\rho}, \tag{8}$$

which in agreement with (1) would take the form

$$\frac{\dot{a}}{a} = -\frac{\dot{T}}{T} - \frac{1}{4}\frac{\dot{N}(T)}{N(T)}. \tag{9}$$

Using the condition of adiabatic expansion of the Universe (6), we can find

$$\frac{\dot{a}}{a} = -\frac{1}{3}\frac{\dot{s}}{s}. \tag{10}$$

Substituting into (10) the expression for specific entropy (2), we obtain

$$\frac{\dot{a}}{a} = -\frac{\dot{T}}{T} - \frac{1}{3}\frac{\dot{N}(T)}{N(T)}. \tag{11}$$

Comparing (9) and (11), we come to the conclusion that $\dot{N}(T) = 0$. So the relation between the scaling factor and temperature should have the form

$$\frac{\dot{a}}{a} = -\frac{\dot{T}}{T}. \tag{12}$$

Substituting this equation into eq. (7), we obtain one of the dynamic equations of the evolution of the Universe:

$$\left(\frac{\dot{T}}{T}\right)^2 + \epsilon(T)T^2 = \frac{4\pi^3}{45}GN(T)T^4. \tag{13}$$

To write the second required equation, let us multiply both parts of the eq. (2) by a^3, express $\dfrac{1}{a^2T^2}$ and substituting it into the auxiliary eq. (3), we obtain the equation

$$\epsilon(T) = k\left[\frac{2\pi^2}{45}\frac{N(T)}{S}\right]^{2/3}, \tag{14}$$

where $S \equiv sa^3$ is the total entropy in the volume defined by the radius of curvature a. We need to note that N, S hence ϵ are constant in the considered temperature (or time) range (however a is not constant) due to the particle counting by allowed particles degrees of freedom thresholds.

As the result, the dynamic equations of the evolution of the Universe in terms of temperature and entropy are the equations (13) and (14).

7. Trapped electron

Consider an isolated conducting sphere of radius R carrying the total charge Q. At the distance $a > R$ from its center, there is a point charge q ($qQ > 0$). Find potential of the system $\varphi(\vec{r})$ and the force $\overrightarrow{F}(a)$ acting on the point charge. Analyze the limit $\lim\limits_{a\to R+0} F(a)$, explain the obtained result.

SOLUTION

The present problem can be solved using method of image charges.

It is known that for any two point electric charges of opposite sign, it is always possible to find such a spherical surface that the resulting potential on it would be zero. Radius of the sphere and the distance from its center to the charges is determined uniquely if the values of the charges and the distance between them are known. So, the system under discussion (the point charge and the conducting sphere) is equivalent to the set of point charges. Thus let us place a charge q_1 on a line connecting the center of the sphere O with the q charge at the distance of d in the direction of the charge q. One more charge, $q_0 = Q - q_1$, we shall place at the O point, see fig. 1.

Let us place the origin of the reference frame also in O point and direct OX axis to the point charge q (leftwards). Let us write the condition that the total potential (due to the charges q and q_1) is zero at points $(R, 0, 0)$ and $(-R, 0, 0)$.

Hence we obtain that $d = R^2/a$, $q_1 = -qR/a$, thus $q_0 = Q + qR/a$.

The total potential inside the full sphere equals q_0/R and outside is given by the expression:

$$\varphi(\vec{r}_0) = \frac{q}{r} + \frac{q_1}{r_1} + \frac{q_0}{r_0}. \tag{1}$$

The same in the Cartesian coordinates:

$$\varphi(\vec{r}_0) = \frac{q}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{R}{a}\frac{q}{\sqrt{(x - \frac{R^2}{a})^2 + y^2 + z^2}} + \frac{Q + q\frac{R}{a}}{\sqrt{x^2 + y^2 + z^2}}, \tag{2}$$

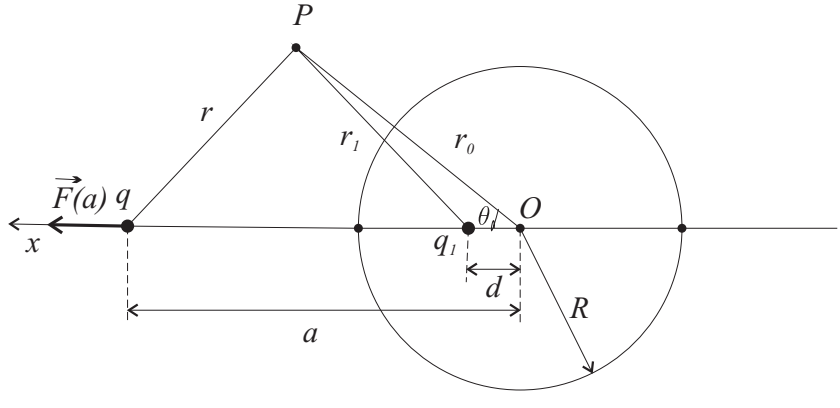


Fig. 1

The force on q acts along axis OX. Its projection is given by

$$F = \frac{qQ}{a^2} + \frac{q^2 R}{a^3}\left(1 - \frac{1}{\left(1 - \frac{R^2}{a^2}\right)^2}\right). \tag{3}$$

It is easy to check that if we represent a as $R + \Delta$, in the limit at $\Delta \to 0$ the expression for the force would become $F \sim -q^2/(2\Delta)^2$ which corresponds to the interaction force of a point charge with non-charged conducting plane.

At $a \to \infty$, $F \to \dfrac{qQ}{a^2}$ (Coulomb force, as expected).

At $a \to R$, no matter how large Q is, even if $Qq > 0$, we have $F \to -\infty$, i. e. the force becomes very strong and attractive!

Let us find the distance at which the effect of attraction starts. Let us write the interaction force (3) in a dimensionless form:

$$f(s) = \frac{\alpha}{s^2} - \frac{2s^2 - 1}{s^3(s^2 - 1)^2}, \tag{4}$$

where

$$\alpha = \frac{Q}{q} > 0, \qquad s = \frac{a}{R} > 1, \qquad f(s) = \frac{R^2}{q^2} F_x(a). \tag{5}$$

The force occurs to be zero at a distance $a = s_0 R$, where $s_0 > 1$ satisfying the equation

$$\frac{2s^2 - 1}{s(s^2 - 1)^2} = \alpha. \tag{6}$$

Function $f(s)$ graphs are shown in fig. 2 at $\alpha = \{2; 1; 0.5\}$. In these cases $s_0 = \{1.43; 1.62; 1.88\}$, and force $f(s)$ reaches its maximum $\{0.43, 0.15, 0.05\}$ when $s = s_{max} = \{1.79, 2.07, 2.46\}$.

This explains why electrons can't escape from metals, even though they are repelled by the other electrons. If an electron manages to escape and gets to a small distance from the surface of metal, the other electrons conspire to bring it back by rearranging themselves in such a way to create a huge image charge which attracts the electron back to the metal with a strong electric force!

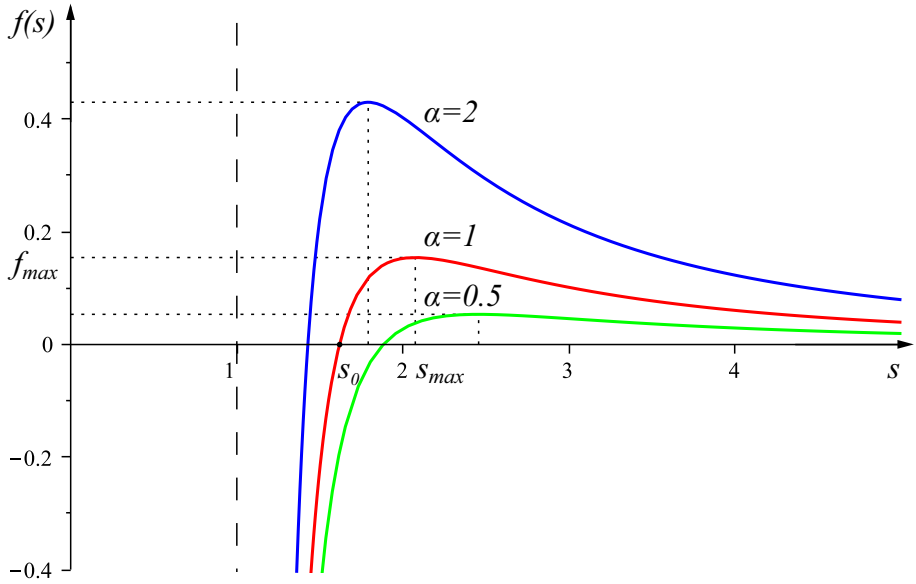


Fig. 2

8. X-sector [3]

Consider the following model of Higgs sector with two doublet scalar fields ϕ_1 and ϕ_2 transformable as $SU(2)$ doublets, with the weak hypercharge generator of $Y_W = 1$, and with each component of the doublet being a complex scalar field. Suppose both fields acquire parallel vacuum averages (vevs) of the type

$$\langle \phi_i \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_i \end{pmatrix} \qquad (i = 1, 2), \tag{1}$$

with the values v_1, v_2 (these vacuum averages lead to gauge bosons mass matrix as in the Standard Model with the replacement [4] $v^2 = v_1^2 + v_2^2$). The most general form of the potential energy function (potential) for a model with two Higgs doublets is rather complicated [5]. However, the model hermitian potential possessing the main properties can be written in the following form:

$$V(\phi_1, \phi_2) = -\mu_1^2(\phi_1^\dagger\phi_1) - \mu_2^2(\phi_2^\dagger\phi_2) - \mu_{12}^2(\phi_1^\dagger\phi_2) - \left(\mu_{12}^2\right)^*(\phi_2^\dagger\phi_1) +$$

$$+\lambda_1(\phi_1^\dagger\phi_1)^2 + \lambda_2(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \frac{\lambda_5}{2}(\phi_1^\dagger\phi_2)^2 + \frac{\lambda_5{}^*}{2}(\phi_2^\dagger\phi_1)^2,$$

where μ_{12}^2 and λ_5 may be the complex numbers.

[3] Some extension of Peskin & Schroeder [6] problem 20.5.

[4] Recall that in the Standard Model (SM), the gauge bosons masses come from the term $|D\phi|^2$ in the Lagrangian, where we set ϕ equal to its vacuum expectation value v.

[5] It is only for reasons of simplicity that the SM contains just a single Higgs doublet. Supersymmetric extensions of the SM typically contain two or more Higgs doublets, and singlets.

(a) Obtain the conditions that for the direction in field space at given configuration of vevs (1) the potential is bounded below at large field values. (The analogue of $\lambda > 0$ for the theory with a single Higgs doublet.)

(b) Find the conditions to impose on the parameters μ and λ, so that the configuration of vacuum averages (1) gives strictly local (locally stable) minimum of this potential.

(c) In the unitary gauge (rotation to canonical form), one linear combination of the upper components ϕ_1 and ϕ_2 nulls, while another becomes a physical field. Show that charged physical Higgs field is of the form:

$$H^+ = \phi_1^+ \sin\beta - \phi_2^+ \cos\beta, \tag{2}$$

where β is defined by the relation

$$\operatorname{tg}\beta = \frac{v_2}{v_1}. \tag{3}$$

(d) Investigate if the CP invariance breaks in the given potential. Substantiate the obtained results.

SOLUTION

(a) First of all, note that $\mu^2_{i=1,2}$ and $\lambda_{i=1\ldots4}$ are all real due to the fact that Lagrangian is Hermitian. But μ^2_{12} and λ_5 may be the complex numbers.

Second, find the condition under which both ϕ_1 and ϕ_2 have parallel non-zero vevs (recall that in the case of a single Higgs doublet the gauge symmetry allows the vacuum expectation value (vev) to be taken in the form $(0, v)$, with v real). Now use a rotation to make $\langle\phi_1\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}$, where v_1 is real. Note that after we have done this we have used all of our gauge rotation freedom, so the vevs of ϕ_2 are still completely general, i. e. $\langle\phi_2\rangle = \dfrac{1}{\sqrt{2}}\begin{pmatrix} v_2' \\ v_2'' \end{pmatrix}$. So we rewrite the potential in terms of the two (complex) vevs of ϕ_2, and of the v_1. Letting $v_2^2 = {v_2'}^2 + {v_2''}^2$, we find

$$V = F(v_1, v_2) - \frac{\mu_{12}^2}{2} v_1 v_2'' - \frac{(\mu_{12}^2)^*}{2} v_1 v_2''^* + \frac{\lambda_4}{4} v_1^2 (v_2''^* v_2'') + \frac{\lambda_5}{8} v_1^2 v_2''^2 + \frac{{\lambda_5}^*}{8} v_1^2 v_2''^{*2}, \tag{4}$$

where we denote $F(v_1, v_2)$ the function dependent on values v_1 and v_2 only.

First we want to answer two questions: (i) What is the condition that v_2'' is real? (ii) What is the condition that v_2' is zero? If we enforce these conditions, it ensures that the vevs take the form of (1), where both v_1 and v_2 are real. This is what it means for the vevs to be 'aligned'. So, how to find these conditions? It becomes clear if we rewrite $v_2'' \equiv a e^{i\theta}$. The potential written in terms of a and θ is given by:

$$V(a,\theta) = (\text{Stuff not dependent on } \theta \text{ or } a) - \mathrm{Re}\mu_{12}^2 v_1 a \cos\theta + \mathrm{Im}\mu_{12}^2 v_1 a \sin\theta +$$

$$+ \frac{1}{4}\lambda_4 v_1^2 a^2 + \frac{1}{4}\mathrm{Re}\lambda_5 v_1^2 a^2 \cos 2\theta - \frac{1}{4}\mathrm{Im}\lambda_5 v_1^2 a^2 \sin 2\theta. \tag{5}$$

The reality of v_2'' is ensured by forcing $\theta = 0, \pi$. We can see that $\theta = 0$ or π will be a stable minimum of the potential if the second derivative of expression (5) is positive at these values of phase θ, i. e.

$$\mathrm{Re}\lambda_5 \; - \; \mathrm{Re}\mu_{12}^2 \frac{1}{v_1 a} < 0, \tag{6}$$

and in the limit of large field values

$$\mathrm{Re}\lambda_5 < 0. \tag{7}$$

In this case the Eqn. (5) will be minimized for $\cos 2\theta = 1$. We must show that $v_2' = 0$. Recall that the sum $v_2'^2 + v_2''^2$ is fixed to be v_2^2. So if we can arrange the potential so that it is energetically advantageous for all the vevs to go into $v_2''^2$, we are done. This is equivalent to saying that we want $V(a, \theta)$ to be minimized for $a \to \infty$. This is accomplished if

$$\lambda_4 + \lambda_5 < 0. \tag{8}$$

So, together, eqns. (7), (8) guarantee that the vacuum expectation values can be aligned.

(b) Now what is still required, is to show that this is a stable minimum. Using aligned forms for the vevs, rewrite the potential in terms of v_1 and v_2. What additional conditions on the parameters are necessary to guarantee a stable minimum? Stability is equivalent to saying that there is a positive mass squared for fluctuations about the minimum. In other words, we examine the mass matrix Hessian

$$H = \left(\begin{array}{cc} \dfrac{\partial^2 V(v_1, v_2)}{\partial v_1^2} & \dfrac{\partial^2 V(v_1, v_2)}{\partial v_1 \partial v_2} \\ \dfrac{\partial^2 V(v_1, v_2)}{\partial v_2 \partial v_1} & \dfrac{\partial^2 V(v_1, v_2)}{\partial v_2^2} \end{array} \right)_{in\ minimum}. \tag{9}$$

In order to have a stable minimum the matrix of second derivatives needs to be positively definite. This does not mean that all second derivatives need to be positive. Both eigenvalues, e_i, need to be positive to ensure the stability of the minimum. The above matrix is to be evaluated at the minimum, setting to zero the derivatives evaluated at the vevs (i. e. where $\dfrac{\partial V}{\partial v_1} = \dfrac{\partial V}{\partial v_2} = 0$). Since invariant $\mathrm{Tr}(H) = e_1 + e_2$ and $\mathrm{Det}(H) = e_1 e_2$, we can simply require

$$\frac{\partial^2 V(v_1, v_2)}{\partial v_1^2} + \frac{\partial^2 V(v_1, v_2)}{\partial v_2^2} > 0, \quad \text{and} \quad \mathrm{Det}(H) \geqslant 0.$$

Straight-forward algebra gives:

$$H = \left(\begin{array}{cc} 2\lambda_1 v_1^2 + \mathtt{Re}\mu_{12}^2 \dfrac{v_2}{v_1} & -\mathtt{Re}\mu_{12}^2 + \lambda_{345} v_1 v_2 \\ -\mathtt{Re}\mu_{12}^2 + \lambda_{345} v_1 v_2 & 2\lambda_2 v_2^2 + \mathtt{Re}\mu_{12}^2 \dfrac{v_1}{v_2} \end{array} \right). \tag{10}$$

We denote $\lambda_{345} = \lambda_3 + \lambda_4 + \mathtt{Re}\lambda_5$.

This shows that stability conditions are equivalent to:

$$2\lambda_1 v_1^2 + 2\lambda_2 v_2^2 + \left(\frac{v_2}{v_1} + \frac{v_1}{v_2}\right) \mathtt{Re}\mu_{12}^2 > 0, \tag{11}$$

and

$$4\lambda_1\lambda_2 + 2\lambda_1 \frac{v_1}{v_2^3}\mathtt{Re}\mu_{12}^2 + 2\lambda_2 \frac{v_2}{v_1^3}\mathtt{Re}\mu_{12}^2 \geqslant (\lambda_{345})^2 - 2\lambda_{345}\mathtt{Re}\mu_{12}^2, \tag{12}$$

by taking first derivatives with respect to all the scalar fields, and setting them equal to zero (i. e. in minimum).

In particular case $\mathtt{Re}\mu_{12}^2 = 0$ we see simply

$$4\lambda_1\lambda_2 \geqslant (\lambda_{345})^2, \quad \text{and} \quad \lambda_1 > 0,\ \lambda_2 > 0. \tag{13}$$

In addition, we will requite concavity, which is implied to be positive, which assures us that we're at a minimum and not at a saddle point. The tricky part of the problem is to show that indeed it is possible to have the vevs parallel, i. e. of the form of (1). [6] To do this, the best way is to use the $SU(2)$ rotation to force the vev of ϕ_1 to have the right form. Then all we need to show is that the potential is minimized (i. e. the appropriate derivatives satisfy the conditions stated above) when ϕ_2 takes the right form.

Thus the conditions for a local stable minimum of the potential in this problem are (7), (8), (11), (12), see also [7].

(c) In the SM, there are 4 degrees of freedom in the Higgs doublet, three of which are consumed by the W^+, W^- and Z^0. When there are 2 Higgs doublets, they contain in total 8 degrees of freedom, so the 5 remaining after goldstones are consumed. Now, the vev given breaks the $SU(2) \times U(1)_Y$ symmetry to $U(1)_{em}$, therefore in general there would be 3 goldstone bosons, and 5 physical Higgs fields. Two of these remaining degrees of freedom are charged, and three are neutral. The task in this problem is to determine which two charged degrees of freedom are eaten, and which two charged degrees of freedom remain. The simplest way to do this is to consider the components of the two Higgs doublets as being part of a larger vector. An orthogonal transformation will rotate the different components amongst themselves. In particular, we can find the basis where the vev is entirely in one neutral component. The charged piece associated with this neutral component is the would-be goldstone boson that is eaten. The charged Higgs is the piece which is orthogonal to this goldstone boson that is eaten. So, considering the neutral components (which have the vevs), we have:

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{14}$$

[6] Thus, in fact, condition (6) is equivalent to positive sign of squared mass for third Higgs (pseudoscalar) boson $m_3^2 = -v_2^2 \mathrm{Re}\lambda_5 + \mathrm{Re}\mu_{12}^2 \dfrac{v_2}{v_1}$, and condition (8) at large field values is equivalent to positive sign of squared mass for charge Higgs boson $m_{H^\pm}^2 = -\dfrac{v_2^2}{2}(\lambda_4 + \mathrm{Re}\lambda_5) + \mathrm{Re}\mu_{12}^2 \dfrac{v_2}{v_1}$.

So, putting all of the vacuum in one Higgs field[7] (say ϕ'_1), we get

$$\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{15}$$

It follows from (15) that $\mathrm{tg}\beta = \dfrac{v_2}{v_1}$. So, the whole vev lives in the ϕ'_1 field, which means that the charged component of ϕ'_1 is the Goldstone Boson eaten by the W. That means $\phi'_1 = \phi_1 \cos\beta + \phi_2 \sin\beta$ is the Goldstone, and

$$\phi'_2 = H^+ = \phi_1^+ \sin\beta - \phi_2^+ \cos\beta \tag{16}$$

is the physical charged Higgs field.

(d) In the models with two doublets of scalar fields CP invariance can be violated by the terms of the potential containing $(\phi_1^\dagger \phi_2)$ or $(\phi_2^\dagger \phi_1)$ with the complex parameters μ_{12}^2 and λ_5. In the case of real parameters CP invariance is not broken.

9. By the cradle of LHC [8]

Generally, it is possible to describe a scattering experiment in the following way (see fig. 3):
1) sufficiently wide uniform beam of particles is prepared so that it is possible to assume the momentum of each particle be equal to $\vec{p}_0 = \hbar\vec{k}$, where $\vec{k}$ is wave vector, $\hbar$ – Planck constant;
2) this beam of particles is directed to stationary target consisting of identical particles;
3) at certain distance from the target, products of reaction of the particles from the beam with the particles forming the target, are registered at different angles.

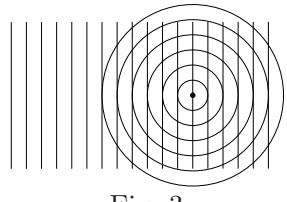


Fig. 3

Thus at sufficiently large distances from the target the wave function of the particles is the superposition of plane incident wave $\psi_{||||} = \exp\{i\vec{k}\ \vec{r}\}$ and spherical scattered wave $\psi_{\odot} = \exp\{ikr\}/r$. Here, $\vec{r} = x\vec{\mathrm{i}} + y\vec{\mathrm{j}} + z\vec{\mathrm{k}}$, $r = |\vec{r}|$.

(a) Calculate flux density of the probability $\vec{j} = \dfrac{\hbar}{2\,m\,i}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$ for the wave functions $\psi_{\rightsquigarrow} = e^{i\vec{k}\cdot\vec{x}}$ and $\psi_{\odot}$. Here, m is the mass of the particle, and $\vec{x} = \vec{i}x$.

(b) Illustrate the obtained result with the help of the graph: draw the pattern of the vector field $\vec{j}$ (lines of the $\vec{j}$ vector) in both cases. To build such graph, use any available computer software suitable for building graphics (plots).

(c) Prove that $\vec{\nabla}\cdot\vec{j}_{\rightsquigarrow} = 0$, $\quad \vec{\nabla}\cdot\vec{j}_{\odot} \sim \delta(\vec{r})$.

[7] Once one goes to the proper basis for describing the Higgs mechanism, there is really only one doublet (in the above case it is ϕ'_1) that acts as the Higgs and has three of four degrees of freedom that are eaten. So, we can say that it is a model with "two-complex scalars".

[8] Large Hadron Collider.

SOLUTION

(a)

$$\vec{j}_{\rightsquigarrow} = \frac{\hbar}{2mi}(\psi^*_{\rightsquigarrow}\vec{\nabla}\psi_{\rightsquigarrow} - \psi_{\rightsquigarrow}\vec{\nabla}\psi^*_{\rightsquigarrow}) = \frac{\hbar}{2mi}\,\vec{\imath}\,[e^{-i\vec{k}\vec{x}}\;ik_x\;e^{i\vec{k}\vec{x}} - e^{i\vec{k}\vec{x}}(-ik_x)\;e^{-i\vec{k}\vec{x}}\;] = \vec{\imath}\,\frac{\hbar k_x}{m},$$

$$\vec{j}_{\odot} = \frac{\hbar}{2mi}(\psi^*_{\odot}\,\vec{\nabla}\psi_{\odot} - \psi_{\odot}\vec{\nabla}\psi^*_{\odot}) =$$

$$= \frac{\hbar}{2mi}\;\vec{n}_r\Big[\frac{e^{-ikr}}{r}\;\frac{d}{dr}\;\frac{e^{ikr}}{r} - \Big(\frac{d}{dr}\;\frac{e^{-ikr}}{r}\Big)\;\frac{e^{ikr}}{r}\Big] = \frac{\hbar k}{m}\frac{\vec{n}_r}{r^2}.$$

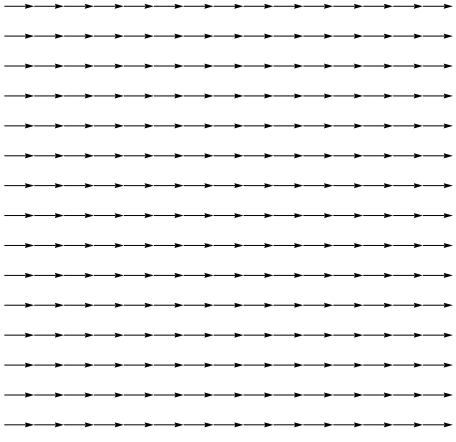
Here, $\vec{n}_r = \vec{r}/r$.

(b)

The field in both cases is represented by a graph in x–y plane at $z = 0$ (in Mathematica realization, see fig. 4).

```
In[1]:= Needs["VectorFieldPlots`"];
In[2]:= VectorFieldPlot[{1, 0}, {x, -1, 1}, {y, -1, 1}]
```

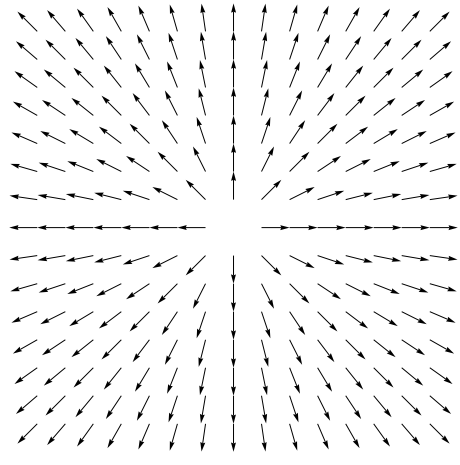
Out[2]=



a

```
In[3]:= VectorFieldPlot[ {x, y} / (x^2 + y^2)^(3/2), {x, -1, 1}, {y, -1, 1},
        ScaleFunction → (1 &) ]
```

Out[3]=



b

Fig. 4

It is possible to see that the spherical wave $\psi_{\odot}$ is diverging from the origin of coordinates.

(c)

Vector field $\vec{j}_{\rightsquigarrow}$ is uniform, $\vec{j}_{\rightsquigarrow} = \vec{\imath}\,\dfrac{\hbar k_x}{m} = \vec{\text{const}}$, so that $\vec{\nabla}\cdot\vec{j}_{\rightsquigarrow} = 0$.

It is possible to represent vector $\vec{j}_{\odot}$ in the form:

$$\vec{j}_{\odot} = \frac{\hbar k}{m}\frac{\vec{n}_r}{r^2} = \frac{\hbar k}{m}\frac{\vec{r}}{r^3} = -\frac{\hbar k}{m}\vec{\nabla}\frac{1}{r}.$$

Hence

$$\vec{\nabla}\cdot\vec{j}_{\odot} = -\frac{\hbar k}{m}\Delta\frac{1}{r} = \frac{\hbar k}{m}4\pi\delta(\vec{r}).$$

10. 'Whipping Top-Toy' from Samara [9]

Rigid ball of the mass m with radius R rests on smooth rigid horizontal surface. Center of mass C of the ball is at the distance of l from its geometric center O. Mass of the material is symmetrically distributed along the volume of the ball relatively to OC axis, and also any plane containing that axis. Moments of inertia of the ball relatively to OC axis and any axis passing the center of mass and perpendicular to OC axis, are equal respectively to J_0 and J. During certain time period, the ball is accelerated around the static vertical axis passing its center O. The moment of time when the action of the "accelerating" forces is finished, is chosen as the time origin. At this moment the ball has the angular velocity $\vec{\omega}_0$, directed vertically up, and OC axis makes some angle ε with the rotation axis of the ball. The center of mass of the ball is lower than its mechanical center (see fig. 5).

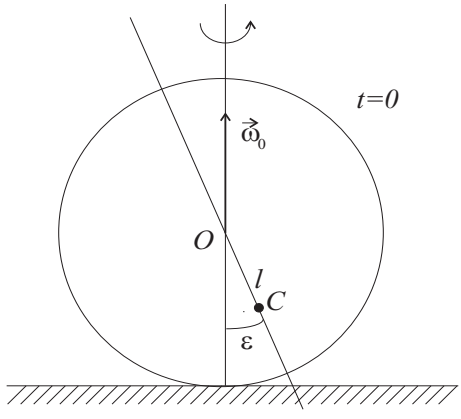


Fig. 5. Rotating ball at time moment $t = 0$

Study the motion of the ball at $t > 0$. Obtain the equation of motion of the ball and find the integrals of motion (conserved values). Show that the center of mass of the ball is lifting up (at a certain relationship between the parameters), and find out, whether it can approach its top position at which the $\overrightarrow{OC}$ vector is directed vertically up.

SOLUTION

In order to describe the motion of the ball we introduce an inertial frame of reference S (with axes X, Y, Z), that is at rest relatively to the horizontal surface, and an noninertial frame S' (with axes X', Y', Z'), that is rigidly bound to the ball. The origin of the frame S coincides with the initial position of the center of the ball, the axis Z is directed vertically and the plane YOZ is chosen so that it contains the axis OC at the initial moment $t = 0$. The origin of the frame S' coincides with the mass center of the ball and the axis Z' – with the axis OC, so that at the moment $t = 0$ the axis Z' makes an angle ε with the axis Z. Additionally, the axis X' at this

[9] Place in Russia where this competition is held and assessed.

moment has the direction similar to that of the horizontal axis X, and the axis Y' belongs to the plane YOZ (see fig. 6).

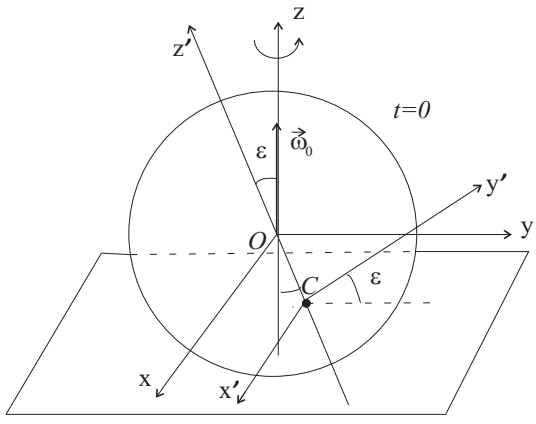


Fig. 6. Coordinate axes and the origins of S, S' at the moment $t = 0$

The reference frames S and S' being chosen, the Euler angles φ, θ, ψ that define the orientation of the ball (and the frame S') relatively to the frame S, are given by the following values at the initial moment:

$$\varphi(0) = 0, \quad \theta(0) = \varepsilon, \quad \psi(0) = 0. \tag{1}$$

Components of the angular velocity $\vec{\omega}$ of the ball in the frames S and S' at any instant of time are given by the cinematic Euler's formulas:

$$\begin{cases} \omega_x = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi, \\ \omega_y = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi, \\ \omega_z = \dot{\varphi} + \dot{\psi}\cos\theta, \end{cases} \tag{2}$$

$$\begin{cases} \omega_{x'} = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\ \omega_{y'} = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \omega_{z'} = \dot{\varphi}\cos\theta + \dot{\psi}. \end{cases} \tag{3}$$

For the initial values of the components of the angular velocity in S the equations (2), with the account of (1), give

$$\begin{cases} \omega_x(0) = \dot{\theta}(0), \\ \omega_y(0) = -\dot{\psi}(0)\sin\varepsilon, \\ \omega_z(0) = \dot{\varphi}(0) + \dot{\psi}(0)\cos\varepsilon. \end{cases} \tag{4}$$

On the other hand, it is known that the initial angular velocity $\vec{\omega}(0) = \vec{\omega}_0$ has the same direction as the axis Z and, hence,

$$\omega_x(0) = \omega_y(0) = 0, \;\; \omega_z(0) = \omega_0. \tag{5}$$

Comparing (4) and (5), one finds the initial values for the time derivatives of the Euler angles:

$$\dot{\varphi}(0) = \omega_0, \;\; \dot{\theta}(0) = 0, \;\; \dot{\psi}(0) = 0. \tag{6}$$

As seen from the pictures, the cartesian coordinates x_m, y_m, z_m of the mass center of the ball in S at the initial moment of time have the following values:

$$x_m(0) = 0, \;\; y_m(0) = l\sin\varepsilon, \;\; z_m(0) = -l\cos\varepsilon. \tag{7}$$

For $t < 0$ (until the rotating forces action has stopped) the center of mass of the ball moves in the horizontal plane along the circle of the radius $l\sin\varepsilon$ with the axis Z passing through its center. Therefore, the velocity $\vec{v}_m$ of the mass center of the ball in the frame S at the initial instant of time is given by

$$\vec{v}_m(0) = [\vec{\omega}_0 \; \vec{r}_m(0)], \tag{8}$$

where $\vec{r}_m(0)$ is the initial value of the radius-vector $\vec{r}$ of the mass center of the ball in S (that is the vector with the components $x_m(0)$, $y_m(0)$, $z_m(0)$). From (5), (7) and (8) it follows that the initial velocity of the mass center of the ball in S is directed opposite to the axis X and is equal to

$$v_m(0) = \omega_0 l\sin\varepsilon, \tag{9}$$

or

$$\dot{x}_m(0) = -\omega_0 l\sin\varepsilon, \;\; \dot{y}_m(0) = \dot{z}_m(0) = 0. \tag{10}$$

The equations of motion of the ball for $t > 0$ can be derived as the Lagrange equations of the 2-nd kind. The ball is a mechanical system with five degrees of freedom. One can use the Euler angles φ, θ, ψ and coordinates x_m, y_m of the mass center in S as the independent generalized coordinates of the ball, the coordinate z_m at any instant of time being defined by

$$z_m = -l\cos\theta. \tag{11}$$

The Lagrange function has the form

$$L = T - U, \tag{12}$$

where T is the kinetic energy of the ball and U is its gravitational potential energy (these values are defined in the frame S).

The kinetic energy of the ball may be expressed in the form

$$T = \frac{mv_m^2}{2} + \tau, \tag{13}$$

where τ is its kinetic energy of rotation. As the axes x', y', z' are directed along the principal axes of inertia of the ball, then it follows

$$\tau = \frac{1}{2}(J_{x'}\omega_{x'}^2 + J_{y'}\omega_{y'}^2 + J_{z'}\omega_{z'}^2), \tag{14}$$

where $J_{x'}$, $J_{y'}$, $J_{z'}$ are the moments of inertia of the ball relatively to the axes x', y', z', with, according to the problem statement,

$$J_{x'} = J_{y'} = J, \;\; J_{z'} = J_0. \tag{15}$$

Inserting the expressions (3) and (15) into (14) one gets

$$\tau = \frac{J}{2}(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{J_0}{2}(\dot{\varphi}\cos\theta + \dot{\psi})^2. \tag{16}$$

The expression for the velocity's square of the mass center of the ball with the account of (11) gets the form

$$v_m^2 = \dot{x}_m^2 + \dot{y}_m^2 + l^2 \sin^2\theta\dot{\theta}^2. \tag{17}$$

From (13), (16) and (17) it follows that

$$T = \frac{m}{2}(\dot{x}_m^2 + \dot{y}_m^2) + \frac{m}{2}l^2\sin^2\theta\dot{\theta}^2 + \frac{J}{2}(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2) +$$

$$+ \frac{J_0}{2}\left(\dot{\varphi}^2\cos^2\theta + 2\dot{\varphi}\dot{\psi}\cos\theta + \dot{\psi}^2\right). \tag{18}$$

The potential energy of the ball with the account of (11) is given by the expression

$$U = -mgl\cos\theta. \tag{19}$$

Inserting the equations (18) and (19) into (12) one gets the following explicit expression for the Lagrange function of the ball:

$$L = \frac{m}{2}(\dot{x}_m^2 + \dot{y}_m^2) + \frac{1}{2}(ml^2\sin^2\theta + J)\,\dot{\theta}^2 + \frac{1}{2}(J\sin^2\theta + J_0\cos^2\theta)\dot{\varphi}^2 +$$

$$+ \frac{J_0}{2}\dot{\psi}^2 + J_0\cos\theta\dot{\varphi}\dot{\psi} + mgl\cos\theta. \tag{20}$$

It is known that the Lagrange equations of the 2-nd kind for a system with ideal holonomic constraints and S degrees of freedom under the absence of dissipative forces have the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \;\; (j = 1, 2, ..., S), \tag{21}$$

where $q_j (j = 1, 2, ..., S)$ are independent generalized coordinates of the system.

If q_j is a cyclic coordinate of the system (that is, the Lagrange function doesn't depend on it), then the corresponding generalized momentum $p_j = \dfrac{\partial L}{\partial \dot{q}_j}$ is the integral of motion.

According to (20), x_m, y_m, φ, ψ are cyclic coordinates of the ball. Therefore, the generalized momentums

$$\frac{\partial L}{\partial \dot{x}_m} = m\dot{x}_m, \tag{22}$$

$$\frac{\partial L}{\partial \dot{y}_m} = m\dot{y}_m, \tag{23}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = J \sin^2\theta\dot{\varphi} + J_0 \cos\theta(\cos\theta\dot{\varphi} + \dot{\psi}), \tag{24}$$

$$\frac{\partial L}{\partial \dot{\psi}} = J_0(\cos\theta\dot{\varphi} + \dot{\psi}) \tag{25}$$

are the integrals of motion of the ball that are constant and are defined by the initial conditions (1), (6), (7) and (10). Writing down the corresponding conservation laws, one gets the following system of the 1-st order differential equations:

$$\dot{x}_m = -\omega_0 l \sin\varepsilon, \tag{26}$$

$$\dot{y}_m = 0, \tag{27}$$

$$J\dot{\varphi}\sin^2\theta + J_0\omega_0 \cos\varepsilon\cos\theta = \omega_0(J\sin^2\varepsilon + J_0\cos^2\varepsilon), \tag{28}$$

$$\dot{\varphi}\cos\theta + \dot{\psi} = \omega_0 \cos\varepsilon \tag{29}$$

(when deriving the equation (28), that presents the conservation law for the generalized momentum $p_\varphi = \dfrac{\partial L}{\partial \dot{\varphi}}$, the equation (29) was used).

From (26) and (27) and the initial conditions (7) it follows that

$$x_m = -\omega_0 t l \sin\varepsilon, \tag{30}$$

$$y_m = l \sin\varepsilon. \tag{31}$$

Hence, the mass center of the ball moves in the vertical plane that is normal to the axis y and that passes through the initial position of the mass center. The horizontal component of the ball's velocity stays constant and equal to the initial velocity $\vec{v}_m(0)$.

Note that the equations (26) – (28) may be also derived from the conservation laws for the components of momentum and angular momentum of the mechanical system in an inertial frame. In order to do this one has to consider the directions of external forces, applied to the ball. These forces are the reaction force $\vec{N}$ that is directed vertically up and the gravitational forces that are applied to every element of the ball and are directed vertically down, their resultant force $m\vec{g}$ being applied to the mass center of the ball.

As the sum of the external forces that are applied to the ball, is directed vertically, the X and Y components of the ball's momentum

$$\vec{P} = m\vec{v}_m \tag{32}$$

are conserved for $t \geq 0$. The corresponding conservation laws, with the account of the initial conditions (10), give the equations (26) and (27).

The angular momentum $\vec{M}$ of the ball in the frame S may be presented in the form

$$\vec{M} = [\vec{r}\vec{P}] + \vec{\mu}, \tag{33}$$

where $\vec{\mu}$ is the intrinsic angular momentum momentum of the ball (that is the angular momentum of the ball in the mass center reference frame that moves translatory to S). The components of $\vec{\mu}$ in S' are defined by

$$\mu_{x'} = J\omega_{x'}, \;\; \mu_{y'} = J\omega_{y'}, \;\; \mu_{z'} = J_0\omega_{z'}. \tag{34}$$

As the moments of all external forces in S that are acting on the ball are directed horizontally for $t \geq 0$, the projection M_z of the angular momentum on the axis Z is the integral of motion. Taking into account (30), (31), (11), (26) and (27), it easy to check that Z-component of the vector $[\vec{r}\vec{P}]$ has the constant value equal to $m\omega_0 l^2 \sin^2\varepsilon$. Hence, with the account of (33), it follows that

$$M_z = m\omega_0 l^2 \sin^2\varepsilon + \mu_z, \tag{35}$$

or that Z-component of the ball's rotate momentum μ_z is an integral of motion.

Using the well-known linear transformation law for the components of an arbitrary vector for a frame rotation and expressing all the coefficients of such transformation as the functions of Euler angles, one may show that

$$\mu_z = \sin\theta\sin\psi\mu_{x'} + \sin\theta\cos\psi\mu_{y'} + \cos\theta\mu_{z'}. \tag{36}$$

Inserting the equations (34), where the angular velocity components are given by the formulas (3), into (36), one gets

$$\mu_z = (J\sin^2\theta + J_0\cos^2\theta)\dot{\varphi} + J_0\cos\theta\dot{\psi}. \tag{37}$$

Comparing the equations (37) and (24), one sees that μ_z coincides with the generalized momentum p_φ. Hence the conservation law for μ_z, written with the account of the initial conditions (1) and (6) and of the equation (29), is the equation (28).

Inserting the equation (20) into (21) and letting $q_j = \theta$, the Lagrange equation for the ball that corresponds to the generalized coordinate θ is obtained:

$$(ml^2 \sin^2\theta + J)\ddot{\theta} + ml^2 \sin\theta\cos\theta\dot{\theta}^2 + (J_0 - J)\sin\theta\cos\theta\dot{\varphi}^2 + \tag{38}$$

$$+J_0 \sin\theta\dot{\varphi}\dot{\psi} + mgl\sin\theta = 0.$$

Note that the equations (28) and (29) form a system of linear algebraic equations relatively to $\dot{\varphi}$ and $\dot{\psi}$ with the coefficients that depend only on the angle θ. Having solved this system, one gets the dependencies of these generalized velocities on θ:

$$\dot{\varphi} = \frac{\omega_0}{J\sin^2\theta}(J\sin^2\varepsilon + J_0\cos^2\varepsilon - J_0\cos\varepsilon\cos\theta), \tag{39}$$

$$\dot{\psi} = \frac{\omega_0}{J\sin^2\theta}[\cos\varepsilon(J\sin^2\theta + J_0\cos^2\theta) - \cos\theta(J\sin^2\varepsilon + J_0\cos^2\varepsilon)]. \tag{40}$$

After having inserted the expressions (39) and (40) into (38) one may get the nonlinear 2-nd order differential equation that defines the dependency $\theta(t)$. In order to get this dependency one may also use the conservation law for the total mechanical energy of the ball

$$E = T + U. \tag{41}$$

This quantity coincides with the generalized energy of the ball and is conserved due to the absence of both dissipative forces and explicit dependency of the Lagrange function on time.

Setting the explicit expression for the energy of the ball that follows from (18), (19) and (41), equal to its initial value, that is found from (1), (6) and (10), and taking into account the conservation laws (26) and (27), one gets

$$(ml^2\sin^2\theta + J)\dot{\theta}^2 + (J\sin^2\theta + J_0\cos^2\theta)\dot{\varphi}^2 + J_0\dot{\psi}^2 + 2J_0\cos\theta\dot{\varphi}\dot{\psi} - 2mgl\cos\theta =$$

$$= (J\sin^2\varepsilon + J_0\cos^2\varepsilon)\omega_0^2 - 2mgl\cos\varepsilon. \tag{42}$$

Substituting the expressions (39) and (40) into the last equation, one obtains the following nonlinear 1-st order differential equation that defines the dependency $\theta(t)$:

$$\sin^2\theta(1 + \beta\sin^2\theta)\,\dot{\theta}^2 = \omega_0^2\,(\cos\varepsilon - \cos\theta)\Theta(\theta), \tag{43}$$

where $\Theta(\theta)$ is quadratic in $\cos\theta$ and has the form:

$$\Theta(\theta) = \mathfrak{a}_0 + 2\mathfrak{a}_1\cos\theta + \mathfrak{a}_2\cos^2\theta, \tag{44}$$

where $\mathfrak{a}_0 - \mathfrak{a}_2$ are given by:

$$\mathfrak{a}_0 = -2\beta\gamma - \frac{1}{4}\left((\alpha-1)^2\cos 3\varepsilon + (\alpha+1)(3\alpha-1)\cos\varepsilon\right), \tag{45}$$

$$\mathfrak{a}_1 = \frac{1}{4}\left((\alpha^2-1)\cos 2\varepsilon + \alpha^2 + 1\right), \tag{46}$$

$$\mathfrak{a}_2 = 2\beta\gamma, \tag{47}$$

Dimensionless parameters α, β, γ in equations (45) – (47) and equation (43), are related to the known parameters in the following way:

$$\alpha = \frac{J_0}{J}, \tag{48}$$

$$\beta = \frac{ml^2}{J}, \tag{49}$$

$$\gamma = \frac{g}{l\omega_0^2} = \frac{\omega_m^2}{\omega_0^2} \tag{50}$$

(here $\omega_m = \sqrt{g/l}$ is the cyclic frequency of a flat simple pendulum of length l).

The region of motion of the ball top is defined by the sign and the roots of the function $\Theta(\theta)$. In order to investigate them one has to specify the range of the parameters that are present in the function. As $J, J_0 \geq 0$ and as for a particular case of the ball being a rotator the moments of inertia have the values $J_0 = 0, J \neq 0$, we conclude that α is not negative. On the other hand, it is known that none of the principle moments of inertia is bigger than the sum of the other two, which gives $J_0 \leq 2J$. Therefore, we conclude that $0 \leq \alpha \leq 2$. Further, as it is easy to notice, $0 \leq \beta < \infty$ and $0 < \gamma < \infty$.

Taking into account the equations (45) – (47), one finds that:

$$0 \leq \mathfrak{a}_1 \leq 2, \tag{51}$$

$$0 \leq \mathfrak{a}_2 < \infty. \tag{52}$$

Concerning $\mathfrak{a}_0$, one may notice that this parameter is not bound from below and approaches its maximum value with respect to $\beta\gamma$ at $\beta\gamma = 0$. Further analysis shows that for $0 \leq \alpha \leq 2, 0 \leq \varepsilon \leq \pi$ the parameter $\mathfrak{a}_0$ reaches its maximum value, that is equal to $\mathfrak{a}_0 = 4$, at $\alpha = 2, \varepsilon = \pi$. For small ε the parameter $\mathfrak{a}_0$ is maximized with respect to α at $\alpha = 0$ and increases monotonically with ε. Hence, we finally conclude that

$$-\infty < \mathfrak{a}_0 \leq 4. \tag{53}$$

From the form of the equation (43) one can conclude that for $\Theta(\varepsilon) > 0$ the motion of the top will always go on in the area where $\cos\varepsilon \geq \cos\theta$ or, that is, where $\theta \geq \varepsilon$. Correspondingly, for $\Theta(\varepsilon) < 0$ the mass center will always be lower than in its initial position during the motion. Finally, for $\Theta(\varepsilon) = 0$ the motion will be going on at constant θ, that is, the mass center of the top will be moving at constant height (however, this regime is unstable with respect to small variations in α, β, γ). The last result follows from the fact that for $\Theta(\varepsilon) = 0$, as we shall see further, the right-hand side of (43) is negative everywhere except for at $\theta = \varepsilon$.

From the equations (44) and (45) – (47) it follows:

$$\Theta(\varepsilon) = -2\sin^2\varepsilon((\alpha - 1)\cos\varepsilon + \beta\gamma) \tag{54}$$

Hence, the condition that the mass center lifts up during the initial stages of motion is:

$$\Theta(\varepsilon) > 0 \Longleftrightarrow (\alpha - 1)\cos\varepsilon + \beta\gamma < 0 \tag{55}$$

Note, in particular, that for the tops with $\alpha > 1$ the mass center will always be moving downwards under the condition $\varepsilon < \dfrac{\pi}{2}$ irrespectively to the kinematic parameters of the problem.

Remark. Note that in [2] the case for small ε is considered in details and analytical solutions are obtained, and the analysis of trajectories is carried out also.

Let us now study the behavior of the roots of the parabola, defined by the dependency $\Theta(\cos\theta)$. As $\mathfrak{a}_2 \geq 0$, the parabola is open upwards (the case $\mathfrak{a}_2 = 0$ will be understood as the limiting one). The minimum of the parabola corresponds to $\cos\theta$ equal to $z_m \equiv -\dfrac{\mathfrak{a}_1}{\mathfrak{a}_2}$. As from the equations (45) – (47), (48) – (50) it follows that $\mathfrak{a}_1, \mathfrak{a}_2$ contain independent parameters, one concludes that

$$-\infty < z_m \leq 0. \tag{56}$$

Further, from (44) one gets that:

$$\Theta_m \equiv \Theta(z_m) = -\frac{\mathfrak{a}_1^2}{\mathfrak{a}_2} + \mathfrak{a}_0. \tag{57}$$

Substitution of (45) – (47) leads to the expression:

$$\Theta_m = -2\beta\gamma - \frac{(\alpha^2\cos^2\varepsilon + \sin^2\varepsilon)^2}{8\beta\gamma} - \cos\varepsilon(\alpha^2 - \sin^2\varepsilon(1-\alpha)^2). \tag{58}$$

As the parameters $\alpha, \beta\gamma$ may be considered as independent, one can say that

$$\Theta_m \leq -(\alpha^2\cos^2\varepsilon + \sin^2\varepsilon) - \cos\varepsilon(\alpha^2 - \sin^2\varepsilon(1-\alpha)^2). \tag{59}$$

The last expression becomes the equality when $\beta\gamma = \dfrac{1}{4}(\alpha^2\cos^2\varepsilon + \sin^2\varepsilon)$. After elementary transformations one arrives at

$$\Theta_m \leq -2\cos^2\frac{\varepsilon}{2}(1 + (\alpha - 1)\cos\varepsilon)^2. \tag{60}$$

Therefore, it is seen that the minimum of the parabola defined by the dependency $\Theta(\cos\theta)$ is a strictly nonpositive quantity that achieves its maximum value equal to zero, under $\varepsilon = \pi$ (and, simultaneously, $\beta\gamma = \frac{1}{4}\alpha^2$). Hence, the binomial $\Theta(\cos\theta)$ does always have two real roots with respect to $\cos\theta$. Let us introduce the notation z_+, z_- for the bigger and the smaller root correspondingly:

$$z_\pm = -\frac{\mathfrak{a}_1}{\mathfrak{a}_2} \pm \sqrt{(\frac{\mathfrak{a}_1}{\mathfrak{a}_2})^2 - \frac{\mathfrak{a}_0}{\mathfrak{a}_2}} \tag{61}$$

Further, notice that:

$$\left.\frac{\mathrm{d}\Theta(\cos\theta)}{\mathrm{d}\cos\theta}\right|_{\theta=\varepsilon} = 4\beta\gamma\cos\varepsilon + \alpha^2\cos^2\varepsilon + \sin^2\varepsilon. \tag{62}$$

The last equation is strictly nonnegative under $\varepsilon \le \frac{\pi}{2}$. Hence, for a motion that starts from the position[10] where $\varepsilon \le \frac{\pi}{2}$, one can state that the region of motion will be defined by

$$\max[-1, z_+] \le \cos\theta \le \cos\varepsilon \tag{63}$$

when the condition (55) is satisfied and

$$\cos\varepsilon \le \cos\theta \le \min[z_+, 1] \tag{64}$$

when it is not satisfied.

Considering the equations (63) – (64) and the equation (61), it is easy to get the condition that at its highest position the mass center belongs to the same horizontal plane as the center of the ball. In order for this to happen the condition (55) and the equation $z_+ = 0$ must hold, which is equivalent to $\mathfrak{a}_0 = 0$ or

$$8\beta\gamma + \big((\alpha-1)^2\cos 3\varepsilon + (\alpha+1)(3\alpha-1)\cos\varepsilon\big) = 0. \tag{65}$$

Note that the last condition can be satisfied only if it admits real roots for α (the other parameters being fixed) and only if one of the roots is in the range $0 \le \alpha \le 1$. The roots for α may be transformed to the form:

$$\alpha_{1,2} = \frac{1}{\cos^2\varepsilon}(-\sin^2\varepsilon \pm \sqrt{\sin^2\varepsilon - 2\beta\gamma\cos\varepsilon}) \tag{66}$$

The condition of the existence of two real roots implies the inequality

$$\sin^2\varepsilon - 2\beta\gamma\cos\varepsilon \ge 0 \tag{67}$$

[10]Further in the text, the condition $\varepsilon \le \frac{\pi}{2}$ will always be considered true.

The condition $\alpha \geq 0$ implies than only the bigger root may be a physically meaningful value and that the following inequality holds:

$$\sin^2\varepsilon\cos^2\varepsilon - 2\beta\gamma\cos\varepsilon \geq 0, \tag{68}$$

Note that it already includes the inequality (67). The condition $\alpha \leq 1$ leads to the inequality

$$\cos^2\varepsilon + 2\beta\gamma\cos\varepsilon \geq 0, \tag{69}$$

which is satisfied automatically.

We can finally conclude that for a top that was initially at the position where $\varepsilon < \frac{\pi}{2}$, the highest position for its mass center will be $\theta = \frac{\pi}{2}$ if the conditions (66), (55) hold, the necessary condition for the first one being (69).

Let us now study the values of the root z_+. When (55) holds (the top lifts up), it is always true that $z_+ \leq \cos\varepsilon$. The lower bound for z_+ may be derived after inserting the equations (45) – (47) into (61), which gives, after transformations:

$$z_+ = \frac{-\lambda^2 + \sqrt{(\mu+\tau)^2 + \lambda^4 - \tau^2}}{\mu}, \tag{70}$$

where

$$\lambda^2 = \sin^2\varepsilon + \alpha^2\cos^2\varepsilon, \tag{71}$$

$$\tau = \cos\varepsilon(\alpha^2\cos^2\varepsilon + (-1+2\alpha)\sin^2\varepsilon), \tag{72}$$

$$\mu = 4\beta\gamma. \tag{73}$$

As in (70) the parameter μ is independent with respect to the other ones, one can minimize this expression with the account of $\mu \geq 0$. It is easy to show that the expression for z_+ is minimal in the limit $\mu \to 0$, converging to the value

$$z_{+\min} = \frac{\tau}{\lambda^2} \tag{74}$$

Now, considering α as a variable, it is easy to see that the equation $z_{+\min}$ is minimal for $\alpha = 0$ and in this case is equivalent to $z_{+\min}|_{\alpha=0} = -\cos\varepsilon$. Therefore, we conclude that, irrespectively to the parameters of the problem, if the mass center lifts up at the beginning of motion, it is always in the range $[\varepsilon, \pi - \varepsilon]$.

Similarly, for the downwards motion the minimal value for z_+ is $\cos\varepsilon$. The maximum value may be derived from the equation (70). Maximizing this expression with respect to μ, one obtains that z_+ reaches its maximum value that is equal to unity in the limit $\mu \to \infty$. Therefore, one can conclude that for any initial position of the top the parameters of the problem may be chosen in such a way that the top may approach arbitrarily close to the position $\theta = 0$. This result, in particular, corresponds to a simple case $\omega_0 = 0$.

In conclusion, notice that the equation (43) may be easily integrated in elementary functions by the use of the variable change $\tau = \cos\theta$.

11. Laplacian $\boldsymbol{\Delta}$ spectrum on a doughnut

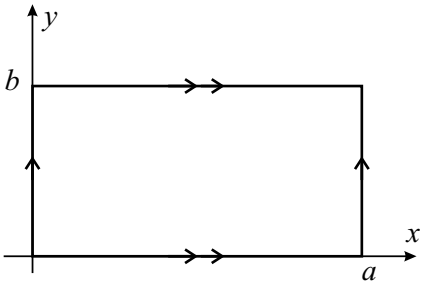


Fig. 7

Consider a torus made of a rectangular block

$$0 \le x \le a, \qquad 0 \le y \le b$$

with glued opposite sides (see fig. 7) where the identical arrows mark the sides to be glued together. For sufficiently big ratio a/b, it is possible to implement such torus nearly without deformations, as the surface of a doughnut in 3D space.

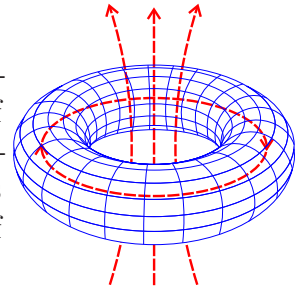


Fig. 8

If inside the doughnut magnetic field with the flux Φ_1 is created, and also magnetic flux Φ_2 is passing through the hole of the doughnut (see fig. 8), then the wave function of the stationary state of the charged particle with the charge e and mass m, on the surface of the torus, is the eigenfunction $\Psi(x,y)$ of the operator

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x} - i\frac{e}{c\hbar}A_x(x,y)\right)^2 + \left(\frac{\partial}{\partial y} - i\frac{e}{c\hbar}A_y(x,y)\right)^2\right], \tag{1}$$

where $\mathbf{A}(x,y)$ is a vector potential, $\hbar$ is Planck constant, and c is speed of light. Function $\Psi(x,y)$ satisfies to periodic boundary conditions:

$$\Psi(x,0) = \Psi(x,b), \qquad \Psi(0,y) = \Psi(a,y), \tag{2}$$

$$\Psi'_y(x,0) = \Psi'_y(x,b), \qquad \Psi'_x(0,y) = \Psi'_x(a,y). \tag{3}$$

Suppose the magnetic field turns to zero on the surface, we can make cuts of the surface and, using gauge transformation, nullify $\mathbf{A}$ on the surface of the torus. Then the wave function would be discontinuous at the cuts (such cuts are possible to make exactly on the border of the rectangle). It is also clear that $|\Psi(x,y)|^2$ should not change.

After the gauge transformation, the task to find stationary states is modified and requires to find eigenfunctions $\psi(x,y)$ of the operator

$$\hat{H}_1 = -\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right], \tag{4}$$

with phase shift (lagging) boundary conditions:

$$\psi(x,0) = e^{i\varphi_1}\psi(x,b), \qquad \psi(0,y) = e^{i\varphi_2}\psi(a,y), \tag{5}$$

$$\psi'_y(x,0) = e^{i\varphi_1}\psi'_y(x,b), \qquad \psi'_x(0,y) = e^{i\varphi_2}\psi'_x(a,y), \tag{6}$$

(a) Find eigenfunctions and eigenvalues of the operator $\hat{H}_1$.

(b) What is the relationship between the fluxes Φ_1, Φ_2 and phase displacements φ_1, φ_2 ?

SOLUTION

Let's make mentioned above cuts of doughnut surface and nullify vector potential $\mathbf{A}'$ with the help of gauge transformation

$$\mathbf{A}'(x,y) = \mathbf{A} - \nabla f(x,y) = 0. \tag{7}$$

Thus the Hamiltonian $\hat{H}_0$ is simplified. In spite of the fact that the wave function would undergo certain changes, it is clear that the probability $|\Psi(x,y)|^2$ of particle location should not be changed by gauge (gradient) transformation, i. e. the new wave function $\psi(x,y)$ differs from the old one $\Psi(x,y)$ only in phase factor [8]:

$$\psi(x,y) = \Psi(x,y)\exp\left(-\frac{ie}{\hbar c}f(x,y)\right), \tag{8}$$

where $f(x,y)$ is the function of gauge transformation, $\psi(x,y)$ is the eigenfunction of the new modified Hamiltonian $\hat{H}_1$, that is in fact Laplace operator.

a) Let us find eigenfunctions $\psi(x,y)$ of the transformed Hamiltonian $\hat{H}_1$. On $\mathbb{R}^2$, eigenfunctions of Laplace operator can be chosen in the form of plane waves $\exp(i\mathbf{kr})$:

$$\psi(x,y) = \exp(ik_x x)\exp(ik_y y). \tag{9}$$

It is clear that any linear combination of eigenfunctions, for which the value $k_x^2 + k_y^2$ is the same, will also be an eigenfunction. Let us find k_x and k_y from the boundary conditions (5), (6):

$$k_x = -\frac{\varphi_2}{a} + \frac{2\pi n_2}{a}, \qquad k_y = -\frac{\varphi_1}{b} + \frac{2\pi n_1}{b}, \tag{10}$$

where $n_1, n_2 \in \mathbb{Z}$.

Eigenvalues of $\hat{H}_1$ are found from the equation $\hat{H}_1\psi(x,y) = E\psi(x,y)$ and have the form:

$$E = \frac{\hbar^2}{2m}\left[\left(-\frac{\varphi_1}{b} + \frac{2\pi n_1}{b}\right)^2 + \left(-\frac{\varphi_2}{a} + \frac{2\pi n_2}{a}\right)^2\right]. \tag{11}$$

Let us prove that we have found the complete basis of the $\hat{H}_1$ eigenfunctions. If the torus is without phase lagging, then we obtain the regular Fourier series. With phase lagging, not quite regular Fourier series is obtained, however it is reducible to the regular one:

$$\psi(x,y) = \exp\left[-i\left(\frac{\varphi_2 x}{a} + \frac{\varphi_1 y}{b}\right)\right]\Psi(x,y).$$

Here, Ψ is a regular periodic function for which the normal Fourier series is written as an expansion using the basis in L_2 for which the completeness has been already proved. Multiplication by $\exp\left[-i\left(\frac{\varphi_2 x}{a}+\frac{\varphi_1 y}{b}\right)\right]$ in terms of space L_2 is unitary transformation.

b) Let us find the relationship between the fluxes of magnetic field Φ_1, Φ_2 and phase lagging φ_1, φ_2. The flux of magnetic field is: $\Phi = \oint\limits_S \mathbf{H}d\mathbf{S} = \oint\limits_S \mathrm{rot}\mathbf{A}d\mathbf{S}$ or, after a transformation using Stokes theorem: $\Phi = \oint\limits_l \mathbf{A}d\mathbf{l}$. Then the fluxes of magnetic field inside the torus and through the hole of the doughnut are respectively equal to:

$$\Phi_1 = \int_0^b A_y(x,y)dy, \tag{12}$$

$$\Phi_2 = \int_0^a A_x(x,y)dx. \tag{13}$$

Recalling expression (7) and substituting it into (12) and (13):

$$\Phi_1 = f(x,b) - f(x,0), \tag{14}$$

$$\Phi_2 = f(a,y) - f(0,y). \tag{15}$$

It is possible to find the differences in values of $f(x,y)$ in these points using expression (8) and boundary conditions (2), (3), (5), (6). As a result we obtain:

$$\Phi_i = \frac{\hbar c}{e}\varphi_i = \Phi_0\frac{\varphi_i}{\pi}, \tag{16}$$

where $i = 1, 2$; Φ_0 is the quantum of magnetic flux. It is necessary to note that in this case the magnetic flux is not quantized.

12. 3D Delta function

Coulomb wave function of the ground state has the form

$$\varphi_c(\vec{p}) = 8\pi\alpha\mu|\varphi_c(r=0)|\varphi_p^2, \qquad |\varphi_c(r=0)|^2 = \frac{\alpha^3\mu^3}{\pi}, \quad \varphi_p = (\vec{p}^2+\mu^2\alpha^2)^{-1}$$

and satisfies Schroedinger equation in the momentum representation.

The values of hyperfine splitting of the ground level of hydrogen-like atom with accuracy up to α^5 found on the basis of quasipotential built from the diagrams of the order of α^2 and higher, are found most easily if to assume

$$\varphi_c(\vec{p}) \approx (2\pi)^3\delta(\vec{p})|\varphi_c(r=0)|. \tag{★}$$

(Infrared singularities in the elements of the amplitude of scattering that appear in the framework of this approach, are normally eliminated by cutting off the value of virtual 3-dimensional momentum.)

Derive the relation (★) by proving the following statements:

(a) $\delta(x) = \lim\limits_{a\to 0} \dfrac{1}{\pi}\dfrac{a}{a^2+x^2}$, $x \in R^1$; (b) $\dfrac{\pi\delta(x)}{2x^2} = \lim\limits_{a\to 0} \dfrac{a}{(a^2+x^2)^2}$, $x \in R^1$;

(c) $\delta(\vec{p}) = \dfrac{\delta(p)}{2\pi p^2}$, $p = |\vec{p}|, p \geqslant 0$; (d) $\delta(\vec{p}) = \lim\limits_{a\to 0} \dfrac{a}{\pi^2(a^2+p^2)^2}$, $p = |\vec{p}|, p \geqslant 0$.

Direction: equations (a) – (d). containing generalized functions, of the class D' should be proved on the space of the basic functions of the class $\mathcal{D}$.

SOLUTION

The above statement (★) is based on the fact that the value of the square of module of the wave function (present in a matrix element) in coordinate space at $r = 0$ has the order [11] of α^3.

Let us prove the (a) statement.

Let $x \in R^1, \varphi(x) \in \mathcal{D}(R^1)$ is the test function. Then

$$\left(\lim_{a\to 0}\frac{a}{\pi}\frac{1}{x^2+a^2}, \varphi(x)\right) = \int\limits_{-\infty}^{\infty} dx \lim_{a\to 0}\frac{a}{\pi}\frac{1}{x^2+a^2}\varphi(x) =$$

$$= \lim_{a\to 0}\frac{a}{\pi}\int\limits_{-\infty}^{\infty}\frac{dx\varphi(x)}{x^2+a^2} = \lim_{a\to 0}\frac{a}{\pi}2\pi i \mathrm{Res}_{x=ia}\frac{\varphi(x)}{x^2+a^2} =$$

$$= \lim_{a\to 0}\frac{a}{\pi}2\pi i\frac{\varphi(ia)}{2ia} = \varphi(0) = (\delta(x), \varphi(x))\,.$$

The first and the last parts are underlined to focus on equality of the functionals hence the generalized functions themselves.

Derivation of the (c) relationship:

$$\delta(\vec{p}) = \delta(p_x)\delta(p_y)\delta(p_z) = (2\pi)^{-3}\int e^{i\vec{k}\vec{p}}d\vec{k} =$$

$$= (2\pi)^{-3}\int\limits_0^{\pi}\int\limits_0^{2\pi}\int\limits_0^{\infty} e^{ikp\cos\theta}k^2dk\sin\theta d\varphi d\theta = (2\pi)^{-2}\int\limits_0^{\infty}\int\limits_{-1}^{1} e^{ikpx}dxk^2dk =$$

$$= (2\pi)^{-2}\int\limits_0^{\infty}\frac{e^{ikpx}}{ikp}\Bigg|_{-1}^{1} k^2dk = \frac{1}{(2\pi)^2 ip}\int\limits_0^{\infty}\left(e^{ikp} - e^{-ikp}\right)kdk = \frac{1}{(2\pi)^2 ip}\int\limits_{-\infty}^{\infty} e^{ikp}kdk =$$

[11] Description of hyperfine splitting is based on computation of quantum distributions being squaresd by the wave function. From the explicit form of the wave function in the momentum representation it turns that the terms of the lowest order in the expansion of a matrix element are α^5.

$$= \frac{1}{(2\pi)^2 ip} \frac{\partial}{i\partial p} \int\limits_{-\infty}^{\infty} e^{ikp} dk = \frac{-1}{2\pi p} \delta'(p) = \frac{\delta(p)}{2\pi p^2}.$$

where the known relation $p\delta'(p) = -\,\delta(p)$ is utilized (the proof is partial integration).

To check (d), we consider the test function $\Phi(\vec{p}) \in \mathcal{D}(R^3)$ and perform the following transformations:

$$\left(\lim_{a\to 0} \frac{a}{\pi^2} \frac{1}{(p^2+a^2)^2}, \Phi(\vec{p}) \right) = \int d\vec{p} \lim_{a\to 0} \frac{a}{\pi^2} \frac{1}{(p^2+a^2)^2} \Phi(\vec{p}) =$$

$$= \int d\Omega \lim_{a\to 0} \frac{a}{\pi^2} \int\limits_0^{\infty} \frac{p^2 \Phi(p,\Theta,\varphi)}{(p^2+a^2)^2} dp =$$

$$= \int d\Omega \lim_{a\to 0} \frac{a}{\pi^2} \left(\frac{1}{2} \int\limits_0^{\infty} \frac{p^2 \Phi(p,\Theta,\varphi)}{(p^2+a^2)^2} \mathrm{d}p - \frac{1}{2} \int\limits_0^{-\infty} \frac{p'^2 \Phi(-p',\Theta,\varphi)}{(p'^2+a^2)^2} dp' \right) =$$

$$\{\text{let} \quad \Phi(p,\Theta,\varphi) = \Phi(-p,\Theta,\varphi)\}$$

$$= \int d\Omega \lim_{a\to 0} \frac{a}{2\pi^2} \int\limits_{-\infty}^{\infty} \frac{p^2 \Phi(p,\Theta,\varphi)}{(p^2+a^2)^2} dp =$$

$$= \int d\Omega \lim_{a\to 0} \frac{a}{2\pi^2} 2\pi i \mathrm{Res}_{p=ia} \frac{p^2 \Phi(p,\Theta,\varphi)}{(p^2+a^2)^2} \quad \text{(residue in the pole of the 2nd order)} =$$

$$= \int d\Omega \lim_{a\to 0} \frac{ai}{\pi} \lim_{p\to ia} \frac{d}{dp} \frac{p^2 \Phi(p,\Theta,\varphi)}{(p+ia)^2} =$$

$$= \int d\Omega \lim_{a\to 0} \frac{ai}{\pi} \lim_{p\to ia} \left\{ \frac{2p\Phi + p^2\Phi'}{(p+ia)^2} - \frac{2p^2\Phi}{(p+ia)^3} \right\} =$$

$$= \int d\Omega \lim_{a\to 0} \frac{ai}{\pi} \left\{ \frac{2ia\Phi(ia,\Theta,\varphi)}{(2ia)^2} + \frac{(ia)^2\Phi'(ia,\Theta,\varphi)}{(2ia)^2} - \frac{2(ia)^2\Phi(ia,\Theta,\varphi)}{(2ia)^3} \right\} =$$

$$= \int d\Omega \lim_{a\to 0} \frac{ai}{\pi} \left\{ \frac{\Phi(ia)}{2ia} + \frac{\Phi'(ia)}{4} - \frac{\Phi(ia)}{4ia} \right\} =$$

$$= \int d\Omega \lim_{a\to 0} \frac{ai}{\pi} \left\{ \frac{\Phi(ia)}{4ia} + \frac{\Phi'(ia)}{4} \right\} = \int d\Omega \frac{\Phi(0)}{4\pi} = \Phi(0) = (\delta(\vec{p}), \Phi(\vec{p}))\,.$$

The truth of the statement (b) follows particularly from the proven (c) and (d). It is also possible to obtain the relationship (b) using the representation of the δ-function from the statement (a).

It is possible to prove the relations (a) – (d) in another way.

Using Sokhotsky's formula

$$\frac{1}{x+i0} = \mathrm{P}\frac{1}{x} - \mathrm{i}\pi\delta(x),$$

which is correct in D', it is easy to derive eq. (a):

$$\begin{aligned}\lim_{a\to 0}\frac{1}{\pi}\frac{a}{a^2+x^2} &= \lim_{a\to 0}\frac{1}{2\mathrm{i}\pi}\left(\frac{1}{x-\mathrm{i}a}-\frac{1}{x+\mathrm{i}a}\right)= \\ &= \frac{1}{2\pi\mathrm{i}}\left(\mathrm{P}\frac{1}{x}+\mathrm{i}\pi\delta(x)-\mathrm{P}\frac{1}{x}+\mathrm{i}\pi\delta(x)\right) = \delta(x).\end{aligned}$$

It is possible to transform eq. (b) into the following form:

$$\delta(x)=\lim_{a\to 0}\frac{2}{\pi}\frac{ax^2}{\left(a^2+x^2\right)^2}. \tag{1}$$

Transforming the right part of the last equation, we obtain by recalling eq. (a):

$$\begin{aligned}\lim_{a\to 0}\frac{2}{\pi}\frac{ax^2}{\left(a^2+x^2\right)^2} &= \lim_{a\to 0}\frac{2}{\pi}\frac{a}{a^2+x^2}\left(1-\frac{a^2}{a^2+x^2}\right)= \\ &= 2\delta(x)-\lim_{a\to 0}\frac{2a^3}{\pi\left(a^2+x^2\right)^2}. \end{aligned}\tag{2}$$

For any function $\varphi(x)\in\mathcal{D}$, the following relations are true:

$$\begin{aligned}\left(\lim_{a\to 0}\frac{2a^3}{\pi\left(a^2+x^2\right)^2},\varphi\right) &= \lim_{a\to 0}\int\limits_{\mathbb{R}}\frac{2a^3\varphi(x)\mathrm{d}x}{\pi\left(a^2+x^2\right)^2}= \\ &= \lim_{a\to 0}\frac{2}{\pi}\int\limits_{\mathbb{R}}\frac{\varphi(at)\mathrm{d}t}{\left(1+t^2\right)^2}= \\ &= \lim_{a\to 0}\frac{2}{\pi}\int\limits_{\mathbb{R}}\frac{[\varphi(at)-\varphi(0)]}{\left(1+t^2\right)^2}\mathrm{d}t+\frac{2\varphi(0)}{\pi}\int\limits_{\mathbb{R}}\frac{\mathrm{d}t}{\left(1+t^2\right)^2}.\end{aligned}$$

The second integral in the last expression can be computed with the help of the transformation $t=\mathrm{tg}\alpha$, $\mathrm{d}t=(t^2+1)\,\mathrm{d}\alpha$:

$$\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}t}{\left(1+t^2\right)^2}=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\alpha\mathrm{d}\alpha=\frac{\pi}{2}.$$

The first integral can be estimated in the following way. Let us split the integral into the two integrals:

$$\begin{aligned}&\lim_{a\to 0}\frac{2}{\pi}\int\limits_{\mathbb{R}}\frac{[\varphi(at)-\varphi(0)]}{\left(1+t^2\right)^2}\mathrm{d}t = \\ &\lim_{a\to 0}\int\limits_{[-A;A]}\frac{2}{\pi}\frac{[\varphi(at)-\varphi(0)]}{\left(1+t^2\right)^2}\mathrm{d}t + \lim_{a\to 0}\int\limits_{\mathbb{R}\setminus[-A;A]}\frac{2}{\pi}\frac{[\varphi(at)-\varphi(0)]}{\left(1+t^2\right)^2}\mathrm{d}t.\end{aligned}$$

The first integral on the right side of the last equation tends to zero at $a \to 0$. For the upper limit of the second integral, it is possible to use the value of

$$\lim_{a\to 0} \int\limits_{\mathbb{R}\setminus[-A;A]} \frac{2}{\pi} \frac{2M}{(1+t^2)^2} \mathrm{d}t = \frac{4M}{\pi} \int\limits_{\mathbb{R}\setminus[-A;A]} \frac{\mathrm{d}t}{(1+t^2)^2},$$

independent of a. At $A \to \infty$, this integral also converges to zero. Thus

$$\left(\lim_{a\to 0} \frac{2a^3}{\pi\left(a^2+x^2\right)^2}, \varphi \right) = \varphi(0).$$

It follows that

$$\lim_{a\to 0} \frac{2a^3}{\pi\left(a^2+x^2\right)^2} = \delta(x).$$

Recalling this result, eq. (2) converges to eq. (1), which proves the validity of eq. (b).

Eq. (c) should take place on the functions from $\mathcal{D}(\mathbb{R}^3)$, i. e. from that, the eq. should follow:

$$\int\limits_{\mathbb{R}^3} \delta(\vec{p})\varphi(\vec{p})\mathrm{d}\vec{p} \;=\; \int\limits_{\mathbb{R}^3} \frac{\delta(p)}{2\pi p^2}\varphi(\vec{p})\mathrm{d}\vec{p}, \tag{3}$$

where $\varphi(\vec{p}) \in \mathcal{D}(\mathbb{R}^3)$. Conversing the right part of this equality, we obtain

$$\int\limits_{\mathbb{R}^3} \frac{\delta(p)}{2\pi p^2}\varphi(\vec{p})\mathrm{d}\vec{p} \;= \int\limits_{\mathbb{R}^3} \frac{\delta(p)}{2\pi p^2}\varphi(\vec{p})p^2\mathrm{d}p\,\mathrm{d}\Omega =$$

$$= \int\limits_{p\geqslant 0} 2\delta(p) \left(\int\limits_{|\vec{p}|=p} \varphi(\vec{p}) \frac{\mathrm{d}\Omega}{4\pi} \right) dp = \int\limits_{\mathbb{R}} \delta(p)\tilde{\varphi}(p)dp = \tilde{\varphi}(0) = \varphi(0).$$

Here, Ω is solid angle, $\tilde{\varphi}(p) = \int\limits_{|\vec{p}|=p} \varphi(\vec{p}) \frac{\mathrm{d}\Omega}{4\pi}$. In the last line, function $\tilde{\varphi}$ was evenly extended to negative p. The integral in the left part of eq. (3) also equals $\varphi(0)$ which proves the validity of that eq. (3) as well as of eq. (c).

Equation (d) automatically follows from (b) and (c):

$$\delta_3(\vec{p}) = \frac{\delta(p)}{2\pi p^2} = \frac{1}{2\pi} \lim_{a\to 0} \frac{2}{\pi} \frac{a}{\left(a^2+p^2\right)^2} = \lim_{a\to 0} \frac{a}{\pi^2\left(a^2+p^2\right)^2}.$$

Let us obtain the final approximate equation (★) for the wave function of the ground state in Coulomb field using the formula $\delta(\vec{p}) = \frac{\delta(p)}{2\pi p^2}$, valid for the case of

spherical symmetry and also using the above proved statements. In the momentum representation this function takes the form:

$$\varphi_c(\vec{p}) = 8\pi\alpha\mu \mid \varphi_c(r=0) \mid \frac{1}{\left(p^2+\alpha^2\mu^2\right)^2},$$

and, using the identity (d), we obtain

$$\lim_{\alpha\to 0}\varphi_c(\vec{p}) = \lim_{\alpha\to 0}\frac{8\pi\alpha\mu|\varphi_c(0)|}{(\vec{p}^2+\alpha^2\mu^2)^2} = 8\pi^3|\varphi_c(r=0)|\delta(\vec{p}).$$

Due to the fact that $\alpha\mu \ll 1$, the function approximately equals its limit at $\alpha\mu \to 0$.

From the form of Coulomb wave function of the ground state it follows that the main contribution into the splitting of the energy levels is due to the momentum from the range that satisfies the condition $\vec{p}^2 \sim \alpha^2\mu^2$. As a result, expansion of the integrand by p/m would be equivalent to an expansion of the whole integral by α (under condition that the integral converges).

13. Heat conduction equation (heat source presents)

The temperature on the ends of thin regular rod is maintained constant and equals zero. Lateral face of the rod is heat-insulated. The frame of reference is defined so that coordinate axis x is oriented along the rod, its ends have the coordinates $x = 0$ and $x = l$.

Thermal diffusivity coefficient of the material of the rod equals a^2.

Find spatial and time distribution $T(x,t)$ $(0 \leqslant x \leqslant l,\ t \geqslant 0)$ of temperature along the rod in two cases:

(a) at time moment $t = 0$ the temperature of the rod is constant, $T(x,0) \equiv T_0$ at $0 < x < l$;

(b) in the center of the rod, point source of intensity Q is switched on at time moment $t = 0$, and $T(x,0) \equiv 0$ at $0 \leq x \leq l$.

SOLUTION

Case (a). In this case, the decision function is defined as the solution of heat conduction equation

$$T_t = a^2 T_{xx}, \tag{1}$$

satisfying supplementary conditions

$$T(0,t) = T(l,t) = 0 \quad (t \geqslant 0), \tag{2}$$

$$T(x,0) = T_0 \quad (0 < x < l). \tag{3}$$

First, let us find the eigenvalues and eigenfunctions of equation (1) using separation of variables (Fourier method).

Representing the decision function in the form

$$T(x,t) = X(x)U(t)$$

and substituting the expression into (1), we obtain

$$X\dot{U} = a^2 U X''.$$

Hence

$$\frac{X''}{X} = \frac{\dot{U}}{a^2 U} = -\lambda^2,$$

$$X'' + \lambda^2 X = 0, \tag{4}$$

$$\dot{U} + a^2\lambda^2 U = 0, \tag{5}$$

where λ^2 =const. Solving eq. (4) and (5) with consideration of boundary conditions (2), we obtain:

$$X_n = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n x}{l}\right) \qquad (0 \leqslant x \leqslant l, \quad \text{orthonormal system of functions}), \tag{6}$$

$$U_n = C_n \exp\left(-\left(\frac{\pi n a}{l}\right)^2 t\right) \qquad (t \geqslant 0), \tag{7}$$

where $n = 1, 2, ...$, and C_n – real coefficients. It is possible to represent the general solution in the form: $T(x,t) = \sum\limits_{n=1}^{\infty} X_n(x)U_n(t)$. Thus

$$T(x,t) = \sum_{n=1}^{\infty} C_n \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n x}{l}\right) \exp\left(-\left(\frac{\pi n a}{l}\right)^2 t\right). \tag{8}$$

According to (8), expansion of the distribution function of initial temperature into series by eigenfunctions (6) takes the form

$$T(x,0) = \sum_{n=1}^{\infty} C_n \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n x}{l}\right),$$

where

$$C_n = \int_0^l T(\xi,0) \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n \xi}{l}\right) d\xi.$$

Recalling initial condition (3)

$$C_n = \sqrt{\frac{2}{l}} T_0 \int_0^l \sin\left(\frac{\pi n \xi}{l}\right) d\xi =$$

$$= \sqrt{\frac{2}{l}} T_0 \frac{l}{\pi n} [-\cos \pi n + 1] = \frac{\sqrt{2l} T_0}{\pi n} \left(1 - (-1)^n\right) =$$

$$= \begin{cases} \dfrac{2\sqrt{2l}}{\pi(2k+1)} T_0, & n = 2k+1, \\ 0, \qquad n = 2k+2 \quad (k = 0, 1, 2, ...). \end{cases} \tag{9}$$

Substituting the expression (9) into (8), we obtain the solution for the case (a):

$$T(x,t) = \frac{4T_0}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \exp\left(-\left(\frac{\pi(2k+1)a}{l}\right)^2 t\right) \sin\left(\frac{\pi(2k+1)x}{l}\right).$$

Case (b). Method I. In this case, the decision function is defined as the solution of heat conductivity equation with the source, i. e. boundary problem with the density of heat generation described by Dirac δ-function:

$$T_t = a^2 T_{xx} + \frac{Q}{c} \delta\left(x - \frac{l}{2}\right), \qquad T(x,0) = 0, \quad T(0,t) = T(l,t) = 0. \tag{10}$$

Here, c is heat capacity of unit length of the thin rod, $Q\delta\left(x - \dfrac{l}{2}\right)$ – intensity of heat generation per unit length.

We shall search for the solution in the form of the sum of a stationary one (ω) with a non-stationary one (v):

$$T(x,t) = \omega(x) + v(x,t).$$

Stationary solution satisfies

$$\omega_{xx} = -\frac{Q}{a^2 c} \delta\left(x - \frac{l}{2}\right), \quad \omega(0) = \omega(l) = 0.$$

And non-stationary solution satisfies

$$v_t = a^2 v_{xx}, \quad v(0,t) = v(l,t) = 0, \quad v(x,0) = -\omega(x).$$

The common solution of stationary equation can be presented in the following form (see [2])

$$\omega = C_1 \left|x - \frac{l}{2}\right| + C_2,$$

where indeed the first constant can be found from differential equation

$$\omega_{xx} = -\frac{Q}{a^2 c}\delta(x - \frac{l}{2}) \;\Rightarrow\; C_1 = -\frac{Q}{2a^2 c},$$

and C_2 from boundary conditions

$$\omega(0) = \omega(l) = 0,$$

exactly

$$-\frac{Q}{2a^2 c}\frac{l}{2} + C_2 = 0 \;\Rightarrow\; C_2 = \frac{Q}{a^2 c}\frac{l}{4}.$$

Finally

$$\omega(x) = -\frac{Q}{2a^2 c}\left|x - \frac{l}{2}\right| + \frac{Q}{a^2 c}\frac{l}{4}.$$

Non-stationary part of the problem is solved using separation of variables, as in the (a) case, only with the boundary condition $v(x,0) = -\omega(x)$. As a result, we obtain $T(x,t) = \omega(x) + v(x,t)$.

Note, that expressions for $\omega(x)$ and $v(x,t)$ can be presented as Fourier series expansions (see details in [2]):

$$\omega(x) = \frac{2Ql}{\pi^2 a^2 c}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{\pi(2k+1)x}{l}\right), \tag{11}$$

$$v(x,t) = -\frac{2Ql}{\pi^2 a^2 c}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{\pi(2k+1)x}{l}\right)\exp\left(-\left(\frac{\pi(2k+1)a}{l}\right)^2 t\right). \tag{12}$$

The answer will be

$$T(x,t) =$$

$$= \frac{2Ql}{\pi^2 a^2 c}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{\pi(2k+1)x}{l}\right)\left[1 - \exp\left(-\left(\frac{\pi(2k+1)a}{l}\right)^2 t\right)\right]. \tag{13}$$

Case (b). Method II. When solving (10), let us expand $T(x,t)$ into Fourier series

$$T(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin\left(\frac{\pi n x}{l}\right),$$

where

$$C_n(t) = \frac{2}{l}\int_0^l T(x,t)\sin\left(\frac{\pi n x}{l}\right)dx.$$

In terms of $C_n(t)$, the equation (10) will take the form (use the expansion of delta-function)

$$\dot{C}_n(t) = -\left(\frac{\pi na}{l}\right)^2 C_n(t) + \frac{2Q}{lc}\sin\left(\frac{\pi n}{2}\right). \tag{14}$$

The general solution of the eq. (14) is the sum of the general solution of the uniform equation and partial solution of the non-uniform equation, i. e.

$$C_n(t) = A_n \exp\left(-\left(\frac{\pi na}{l}\right)^2 t\right) + \frac{2Q}{lc}\sin\left(\frac{\pi n}{2}\right)\left(\frac{\pi na}{l}\right)^{-2}. \tag{15}$$

Recalling the initial condition (10), we obtain

$$C_n(0) = \frac{2}{l}\int\limits_0^l T(x,0)\sin\left(\frac{\pi nx}{l}\right)dx = 0. \tag{16}$$

Considering (16), from (15) it follows that

$$A_n = -\frac{2Q}{lc}\sin\left(\frac{\pi n}{2}\right)\left(\frac{\pi na}{l}\right)^{-2}. \tag{17}$$

Substituting (17) into (15), we obtain

$$C_n(t) = \frac{2Q}{lc}\sin\left(\frac{\pi n}{2}\right)\left(\frac{\pi na}{l}\right)^{-2}\left[1 - \exp\left(-\left(\frac{\pi na}{l}\right)^2 t\right)\right]. \tag{18}$$

Taking into account (18), we can write the solution of eq. (10) in the form

$$T(x,t) = \frac{2Q}{l}\sum_{n=1}^{\infty}\sin\left(\frac{\pi n}{2}\right)\left(\frac{\pi na}{l}\right)^{-2}\left[1 - \exp\left(-\left(\frac{\pi na}{l}\right)^2 t\right)\right]\sin\left(\frac{\pi nx}{l}\right). \tag{19}$$

Because

$$\sin\left(\frac{\pi n}{2}\right) = \begin{cases} (-1)^k, & n = 2k+1, \\ 0, & n = 2k+2, \end{cases}$$

where $k = 0, 1, 2, ...$, the solution (19) coincides with (13).

In the limit $t \to \infty$, the solution (19) would tend to the stationary solution $\omega(x)$, determined by (11).

14. Heat conduction equation with nonlinear add-on

Burgers' equation is a fundamental partial differential equation from fluid mechanics and other areas of applied mathematics. It bears the name of the Dutch physicist Johannes Martinus Burgers (1895 – 1981). For a given velocity of a fluid u and its viscosity coefficient ν, the general form of Burgers' equation has the following form: $v_t + vv_x = \nu v_{xx}$.

Show that it could be linearized by substitution

$$v = -2\nu\frac{\partial}{\partial x}\ln f,$$

and reduced to the heat conductivity equation $f_t = \nu f_{xx}$.

SOLUTION
Substitute the expression

$$v = -2\nu f_x/f \tag{1}$$

into the Burgers equation. The result of such substitution is that the all derivatives in the Burgers equation obtain the form:

$$\begin{gathered} v_t = -2\nu\frac{f_{xt}}{f} + 2\nu\frac{f_x f_t}{f^2}, \qquad v_x = -2\nu\frac{f_{xx}}{f} + 2\nu\frac{f_x^2}{f^2}, \\ v_{xx} = -2\nu\frac{f_{xxx}}{f} + 6\nu\frac{f_{xx}f_x}{f^2} - 4\nu\frac{f_x^3}{f^3}. \end{gathered} \tag{2}$$

Substituting the expressions (2) into the Burgers equation, we obtain

$$-\frac{f_{xt}}{f} + \frac{f_x f_t}{f^2} = \nu\Big(-\frac{f_{xxx}}{f} + \frac{f_{xx}f_x}{f^2}\Big).$$

The obtained equation can be transformed in the following way:

$$\frac{\partial}{\partial x}\Big(\frac{f_t}{f}\Big) = \nu\frac{\partial}{\partial x}\Big(\frac{f_{xx}}{f}\Big).$$

Then, for f we obtain almost the equation of heat conductivity (or diffusion):

$$f(x,t)_t = \nu f(x,t)_{xx} + \mathrm{F}(t)f(x,t),$$

where $\mathrm{F}(t)$ is an arbitrary time function. If $\mathrm{F}(t) = 0$, we really obtain the heat conductivity (or diffusion) equation.

Short reference. Suppose that in a certain region of space all particles are moving along straight lines parallel to the X axis.

Let us designate $v = dx/dt$ – the projection of the medium velocity (being the function of the coordinate of the point x and time t) on the X axis. The equation of free one-dimensional motion of incompressible fluid is written in the form:

$$v_t + vv_x = 0 \tag{3}$$

and, as seen, is non-linear. It has a solution in the form of traveling waves the front of which is becoming more steep with time and as a result the wave breaks. There are many examples of breaking waves from which perhaps the most visual would be

formation of the white caps on the sea surface at strong acceleration of the waves by the wind.

Of course, waves breaking does not always take place. There are some existing factors that stop process of steeping wave fronts.

One of such factors is viscosity. If we add the viscosity term to the equation (3) then we obtain the Burgers equation

$$v_t + vv_x = \nu v_{xx}.$$

Here, ν is the viscosity factor. Within this model, it is possible to describe the waves in which the competition takes place between the two opposite processes, steeping wave fronts due to non-linearity and quenching due to viscosity. As a consequence of such competition, stationary motion can appear.

The point of interest of the Burgers equation is the existence of exact solution built by Hopf [9] and Cole [10]. Transformation leading to linearization of the Burgers equation (recalled in the statement of this problem) is called in literature as Cole–Hopf transformation.

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DATA ON AUTHORS

George Sergeyevitch Beloglazov – candidate of physical and mathematical sciences, associate professor, The University of Dodoma - UDOM, Tanzania; Perm State Pharmaceutical Academy.

Bobrick Alexey Leonidovich – post-graduate student, Faculty of Science, Department of Astronomy and Theoretical Physics, Lund University, Sweden.

Chervon Sergey Viktorovich – doctor of physical and mathematical sciences, professor, Ulyanovsk State Pedagogical University.

Danilyuk Boris Vasilievich – Senior Lecturer, Samara State University.

Dolgopolov Mikhail Vyacheslavovich – candidate of physical and mathematical sciences, associate professor, manager of Samara State University mathematical physics research Laboratory.

Ivanov Mikhail Gennadievich – candidate of physical and mathematical sciences, associate professor, Moscow Institute of Physics and Technology (State University).

Panina Olga Gennadievna – assistant, Samara State Aerospace University.

Petrova Elena Yurevna – student, Samara State University.

Rodionova Irina Nikolaevna – candidate of physical and mathematical sciences, associate professor, Samara State University.

Rykova Elza Nurovna – candidate of physical and mathematical sciences, Senior Lecturer, Samara State University.

Shalaginov Mikhail Yuryevich – post-graduate student, Purdue University, US.

Tsirova Irina Semyonovna – candidate of physical and mathematical sciences, associate professor, Samara State University.

Volovich Igor Vasilievich – corresponding member of the Russian Academy of Sciences, head of mathematical physics department of Steklov Mathematical institute of RAS, scientific supervisor of Samara State University mathematical physics research Laboratory.

Zubarev Alexander Petrovich – candidate of physical and mathematical sciences, leading research fellow of Samara State University mathematical physics research Laboratory.

Annex

Statements of the Problems of the Second International Olympiad on Mathematical and Theoretical Physics «Mathematical Physics»

September, 4 – 17, 2010

www.labmathphys.samsu.ru/eng

1) "Linear-Nonlinear" response. Non-linear Burgers equation $v_t + vv_x = \nu v_{xx}$ can be linearized using Coal–Hopf transformation

$$v = -2\nu \frac{\partial}{\partial x} \ln f. \tag{1}$$

Here, $v(x,t)$ is the solution of Burgers equation, $f(x,t)$ – solution of the heat conduction equation $f_t = \nu f_{xx}$.

For the Burgers equation, the initial condition is given as:

$$v(x,0)\big|_{t=0} = v_0(x), \quad \int\limits_{-\infty}^{\infty} v_0(x)dx < \infty.$$

(a) Using the transformation (1) for $v_0(x)$, find the corresponding function $f_0(x)$ initial condition for the heat conduction equation.

(b) The general solution to the Cauchy problem for the heat conduction equation is known:

$$f(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} f_0(y) \exp\left[-\frac{(x-y)^2}{4\nu t}\right] dy, \quad x, y \in R^1, \quad t \geq 0. \qquad (2)$$

Using transformation (1), obtain the solution $v(x,t)$ of the Burgers equation.

Hints

1. To answer the question (b) use the result obtained in (a) for the present problem.

2. It is convenient to express the answer to (b) using the function

$$\psi(x,t;y) = \int_0^y v_0(x')dx' + \frac{1}{2t}(x-y)^2.$$

(c) For Burgers equation, the initial condition is given:

$$v(x,0) = v_0(x) = \frac{1}{1+(x-5)^2} + \frac{1}{1+(x+5)^2}.$$

Use for this case the solution scheme developed above in paragraphs (a) and (b), find system's response $v(x,t)$. Follow and analyse evolution of the obtained solution in time. Utilize the 'Mathematica' package.

2) Harmony of a flute.

In woodwind and brass musical instruments, the source of the sound is the oscillating column of air. In a pipe, the standing waves emerge. Such vibrations occur at certain eigen frequencies.

Oscillations of pressure in a pipe of length L are described by the wave equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho_0}{\beta}\frac{\partial^2 p}{\partial t^2},$$

where p is the overpressure (relative to the atmospheric), ρ_0 – density of air in the pipe, β – modulus of volume elasticity, x – coordinate along the pipe axis (see fig. 1), t is time.

A specific solution of that equation is the function

$$p(x,t) = (A\cos kx + B\sin kx)\cos\omega t.$$

(a) Find the values of $A,\ B,\ k,\ \omega$ when both ends of the pipe are open, and also the condition

$$p(x = L/2, t = 0) = p_0 > 0$$

is met.

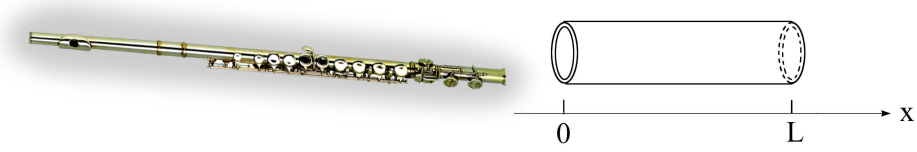


Fig. 1

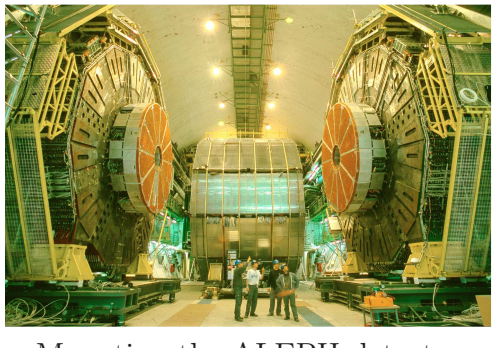
(b) On the basis of the solution obtained, analyse the time evolution of the gas pressure $p(x,t)$ in the pipe. For this purpose, you are encouraged to use the graphics features of the 'Mathematica' software package.

For your information:
1) air density under normal conditions is $\rho = 1,29$ kg/ m^3, and the modulus of volume elasticity is $\beta = 1,01 \times 10^5$Pa;
2) the length of a flute may vary widely, so for illustration, it is possible to choose L=0.5 m.

(c) Illustrate the obtained solution using sound synthesis features of the 'Mathematica' software package. Stipulate an opportunity to hear the fundamental tone and some overtones of the pipe of variable length. How would the tone of the pipe depend on the following parameters: ρ_0, β, L?

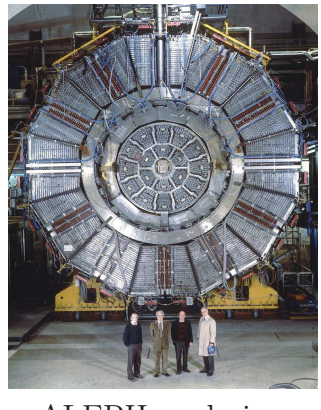
3) From the history of LHC: LEP.

At the end of the XXth century, the colliding beams experiments on electron-positron accelerator have been held in CERN. Such collider is known as the LEP-collider (Large Electron-Positron). The detectors (see figure) recording collisions of particles with anti-particles were placed at the intersections of the colliding beams.



Mounting the ALEPH detector

Fig. 2



ALEPH, end view

Fig. 3

To study the collision pattern, it is required not only to find out which particles are born but also to measure their characteristics with high precision, reconstruct the particles' trajectories, find out their momenta and energies.

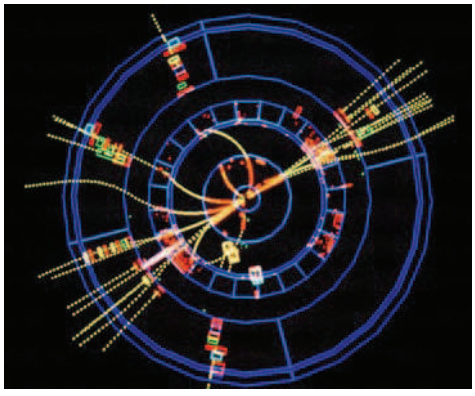


Fig. 4. Curvature of the particles tracks in a magnetic field

Such measurements are held with the aid of various types of detectors that coaxially surround the place of the collision of the particles. In the area of magnetic field, curvature of a trajectory (see fig. 4) enables to find out the momenta of the products of a reaction.

(a)

On the figure, the event of the birth of a neutral K^0– meson (kaon) is shown. The length of its trajectory is 0.1542208 m, the momentum equals $1.197206 \cdot 10^{-18}$ kg m/s (or 2.240160 GeV/c), the speed of meson is $0.976200c$, where c is the speed of light in vacuum. Using these data, find out intrinsic lifetime of K^0– meson, its total and kinetic energies (in GeV).

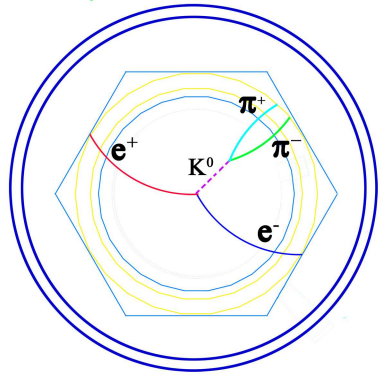


Fig. 5

(b)

In a magnetic field with induction $B = 1.52$ T, K^0– meson decays into π^+ and π^-– mesons with momenta $5.143114 \cdot 10^{-19}$kg m/s and $7.027504 \cdot 10^{-19}$kg m/s, respectively. Analyse maximum possible value of radii of the circles of lateral motion (with respect to $\vec{B}$) of $\pi^{\pm}$-mesons. Also, find the angle of their divergence. The elementary charge $e_0 = 1.6 \cdot 10^{-19}$ Clmb.

4) Virial of gravitational collapse. In classical mechanics of systems executing finite motion, the following relationship takes place:

$$\langle K \rangle = -\frac{1}{2}\Big\langle \sum_i \vec{F}_i \cdot \vec{r}_i \Big\rangle. \tag{3}$$

Here, $\langle K \rangle$ is the mean (for sufficiently long time interval) kinetic energy of the system of point particles defined by radius vectors $\vec{r}_i$ and exposed to the action of the forces $\vec{F}_i$.

A planet revolves around the Sun. Interaction between the planet and the Sun obeys the law of universal gravitation. The mass of the Sun is much larger than the mass of the planet so the heliocentric reference frame can be considered inertial.

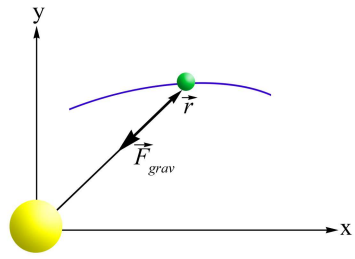


Fig. 6 to the problem 4) (a)

(a)
Obtain the relationship between the mean kinetic $\langle K \rangle$ and mean potential $\langle U \rangle$ energies of the planet directly from the Virial Theorem (3).

(b)
A planet revolves around the Sun along the circular orbit of the radius R. Show that the kinetic K and potential U energies of the planet on its circular orbit are related in the following way

$$K = -\frac{1}{2}U. \qquad (4)$$

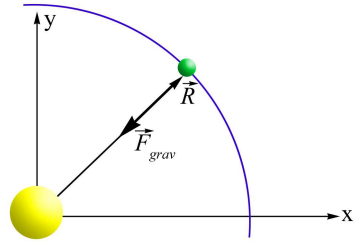


Fig. 7 to problems 4) (b), (c)

(c)
A planet revolves around the Sun along the circular orbit of the radius R. If the mass of the Sun would instantly diminish by 2 times, what will be the trajectory of the planet? What relationship would be given by the Virial Theorem in this case?

5) Waves on Moebius strip. A Moebius strip is a rectangular block $0 \le x \le a$, $0 \le y \le b$, where points with coordinates $(0, y)$ and $(a, b - y)$ are glued together (see fig.).

For sufficiently large ratio a/b, the Moebius strip can be implemented nearly without stretching as a surface with an edge in three-dimensional space.

Let the oscillations of the surface of Moebius strip be described by the wave equation for the function $u(x, y, t)$

$$u_{tt} - \triangle u = 0.$$

The edge of the Moebius strip is free, and hence Neuman's boundary condition is set (see fig. 8)

$$u_y(x, 0, t) = u_y(x, b, t) = 0.$$

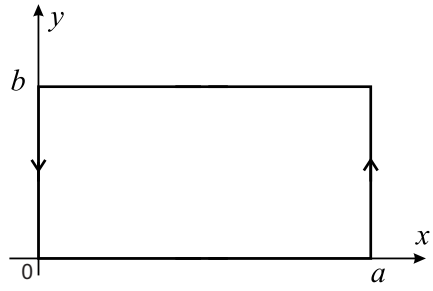


Fig. 8

1) State boundary conditions on the gluing line (points with the coordinates $(0, y)$ and (a, y), $y \in [0, b]$) corresponding to the longitudinal vibrations (u is a small displacement along the surface). Find eigen harmonic oscillations as the solutions of the wave equation with corresponding boundary conditions.

2) State boundary conditions on the gluing line (points with the coordinates $(0, y)$ and (a, y), $y \in [0, b]$) corresponding to the transverse vibrations (u is a small displacement perpendicular to the surface). Find eigen harmonic oscillations as the solutions of the wave equation with corresponding boundary conditions.

6) Collapse of a bubble. Smooth 2-dimensional surface without self-intersections in 3-dimensional space is topologically equivalent to a sphere. At the initial time moment, the surface bounds the volume V. Points of the surface are moving with normally oriented variable velocities. At each time moment, the projection of the velocity on the internal normal equals the Gauss curvature (product of the two main curvatures) of the surface. Let the surface remain smooth during the process of the motion, self intersections do not occur. At certain time moment, the surface collapses into a point. What time will it take the surface to collapse into a point?

7) Random problem. Let ξ_1 and ξ_2 be positive random variables on probability space $\{\Omega, \mathcal{F}, \mathsf{P}\}$, and such that for all real $p \in [a, b]$, $0 < a < b$,

$$\mathsf{E}\,\xi_1^p = \mathsf{E}\,\xi_2^p < \infty.$$

E is denoted as the operator of mathematical expectation value:

$$\mathsf{E}\xi^p := \int\limits_{\Omega} \xi^p \, d\mathsf{P} = \int\limits_{-\infty}^{+\infty} x^p \, dF_\xi(x).$$

Prove that their distribution functions coincide:

$$F_{\xi_1}(x) = \mathsf{P}\{\xi_1 \leqslant x\} = \mathsf{P}\{\xi_2 \leqslant x\} = F_{\xi_2}(x) \text{ for all } x \in \mathbb{R}.$$

8) Maximal domain for a matrix. Find maximal domain in which the Cauchy problem

$$U(x,t)|_{t=0} = T(x), \frac{\partial U}{\partial t}|_{t=0} = N(x), \tag{5}$$

for the system of equations

$$U_{tt} - AU_{xx} = 0, \tag{6}$$

with the matrix $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ has a unique solution for any $x \in (0,1)$.

9) Abel's Analogue. In 1823 Abel has been working on the generalization of the Tautochrone Problem (to find a curve along which a heavy particle moving without friction would reach its lowest position for the same time independing on its initial position). Abel has reached to the equation

$$\int_0^x \frac{f(t)dt}{\sqrt{x-t}} = \varphi(x), \tag{7}$$

where $f(x)$ is the decision function, $\varphi(x)$ – given function.

Solution of the equation has the form

$$f(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}}.$$

In the present problem, it is offered to find a solution of the trigonometric analogue of the equation (7)

$$\varphi(x) = \int_0^x \frac{f(t)dt}{\sqrt{\sin(x-t)}}, 0 < t < x < \frac{\pi}{2}, \tag{8}$$

where $\varphi(x) = \dfrac{1}{\sqrt{\cos x}}$.

10) Problem of energy decomposition. Let $u(x,t) \in C^2(R \times [0,\infty))$ be a solution of the Cauchy initial value problem for one-dimensional wave equation

$$u_{tt} - a^2 u_{xx} = 0$$

in $R \times (0,\infty)$, with initial conditions

$$u(x,0) = g(x),\ u_t(x,0) = h(x),$$

where $g(x), h(x)$ are finite functions.

Kinetic energy $K(t) = \frac{1}{2}\int\limits_{-\infty}^{+\infty} u_t^2(x,t)dx.$

Potential energy $P(t) = \frac{1}{2}\int\limits_{-\infty}^{+\infty} u_x^2(x,t)dx.$

Prove that

a) $K(t) + P(t)a^2$ is constant for any t.

b) $K(t) = P(t)a^2$ for rather large t.

11) Dirac Problem. When deriving so called «Dirac equation» in relativistic quantum mechanics, Dirac has been driven by an idea of «square-rooting» from a second order differential operator.

Find out in terms of square operator of the first order:

a) a wave one-dimensional operator;

б) Laplace operator in R^2.

12) Certain process for a wave equation.

Some process is simulated by a function $u(x,t)$, that satisfies the initial conditions

$$u(x,0) = \left[\begin{array}{ll} \sin^2 \pi x, & 0 \le x \le 1, \\ 0, & x < 0 \text{ and } x > 1, \end{array}\right. \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

It is known that even part of this function $u^r(x,t)$ satisfies the wave equation

$$u^r_{tt} - a^2 u^r_{xx} = 0$$

in half plane $t > 0$. Odd part of this function u^n satisfies the wave equation

$$u^n_{tt} - b^2 u^n_{xx} = 0.$$

Find the distance between x-coordinates at which $u(x,T)$ has minimal values at sufficiently large T.

13) Maximal domain and a square. For the equation

$$u_{xx} + \sqrt{y} u_{xy} = 0 \tag{9}$$

find maximal domain area on the x - y plane, where

$$u(x,x) = \varphi(x), \qquad \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \psi(x), \qquad 0 < x < 1.$$

Show that this domain can be divided into 3 parts by straight linear cuts from which it is possible to make a square block. What will be the area of such square block?

14) Evaluation of the solution of the ultrametric diffusion type of equation with fractional derivative. When solving equations of the ultrametric diffusion type (such equations are related to describing conformation dynamics of compound systems such as biomacromolecules), the solutions are often represented in the form of exponent series. One of such series is presented below:

$$S(t) = \sum_{i=0}^{\infty} a^{-i} E_{\beta}(-b^{-i} t^{\beta}).$$

Here, $E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}$ is Mittag-Leffler function, $0 < \beta \leqslant 1$, and t is time, $S(t)$ – probability of finding the system in definite state groups, $a > 1$, $b > 1$ – certain parameters.

Study asymptotic behavior of the function $S(t)$ at $t \to \infty$ and find its asymptotic evaluation by t-depending elementary functions.

MATHEMATICAL PHYSICS

PROBLEMS AND SOLUTIONS

The Students Training Contest Olympiad
in Mathematical and Theoretical Physics
(On May 21st – 24th, 2010)

Special Issue № 3
of the Series of Proceedings «Modern Problems of Mathematical Physics»

Authors:
G.S. Beloglazov, A.L. Bobrick, S.V. Chervon, B.V. Danilyuk,
M.V. Dolgopolov, M.G. Ivanov, O.G. Panina, E.Yu. Petrova, I.N. Rodionova,
E.N. Rykova, I.S. Tsirova, M.Y. Shalaginov, I.V. Volovich, A.P. Zubarev

Title Editing T.A. Murzinova
Computer Design M.V. Dolgopolov
Art drawings Jy.A. Novikova
Cover Art Design L.N. Zamamykina

Signed for printing: 31.11.2010. Format 70×108/16.
Paper offset. The press offset.
Cond. sheets 5,95; acc.publ. 4,25. Type family: Times New Roman.
Circulation: 100 copies. Order № 300. Samara University Press
1 Academic Pavlov st., Samara, 443011.
Tel. +7 846 334-54-23

View publication stats